Pullbacks of vector bundles and connections

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1 Homomorphisms of vector bundles over possibly different manifolds

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} N$ be vector bundles over manifolds M, N respectively. A *vector-bundle homomorphism* (also called simply a *bundle homomorphism*, *homomorphism*, or, more ambiguously, a *bundle map*[1]) is a smooth map $F : E' \to E$ that carries each fiber of E' linearly into a fiber of E (not necessarily injectively or surjectively). Given any such F and any $p \in N$, the image $\pi(F(E_p))$ is a unique point in M, so we may define a function $f : N \to M$ by $f(p) = \pi(E_p)$, yielding the commutative diagram in Figure 1.

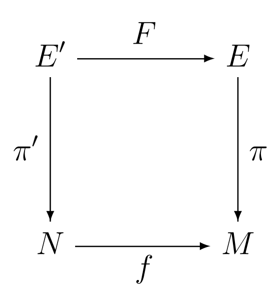

Figure 1: *Homomorphism of vector bundles.* The diagram above commutes. For each $p \in N$, the map $F|_{E'_p} : E'_p \to E_{f(p)}$ is linear.

We say that the bundle map $F : E' \to E$ *covers* the map $f : N \to M$.

[1] The term "bundle map" can be applied to general fiber bundles, not just vector bundles. "Homomorphism" is used only when there is some algebraic structure preserved by a map.

Temporarily letting $s : N \to E'$ denote the zero-section of E' (i.e. $s(p) = 0_{E'_p}$ for all $p \in N$), observe that $f = \pi \circ F \circ s$, a composition of smooth maps. Hence the map $f : N \to M$ covered by the bundle-map F is itself smooth.

The most common vector-bundle homomorphisms are those of *constant rank*, i.e. those for which $\mathrm{rank}(F|_{E'_p})$ is the same for all $p \in N$. Among these, the most important are *monomorphisms* and *epimorphisms*, those bundle homorphisms $F : E' \to E$ such that for all $p \in N$, the linear map $F|_{E'_p} : E'_p \to E_{f(p)}$ is, respectively, injective or surjective. Note that in each of these cases, the covered map f need not be injective or surjective; the injectivity/surjectivity refers purely to the *fiberwise* behavior of F.

The term *isomorphism* (of vector bundles) is author-dependent: all authors require a bundle isomorphism F to carry fibers isomorphically to fibers, but some authors (including me) tend not to use the term *isomorphism* unless, additionally, F covers the identity map (i.e. the case in which $N = M$ and $f = \mathrm{id}_M$).

2 Pullbacks of vector bundles

Informally, we may think of a rank-k vector bundle over a manifold M as a "smoothly parametrized" collection of k-dimensional vector spaces $\{E_q\}_{q\in M}$; the parameter-space is M. The definition of *vector bundle* gives precise meaning to "smoothly parametrized": existence of a vector-bundle atlas for the set $E = \coprod_{q\in M} E_q$.

Given manifolds M and N, a rank-k vector bundle $E \xrightarrow{\pi} M$, and $f : N \to M$ be a smooth map, the collection of vector spaces $\{E_{f(p)}\}_{p\in N}$ is again a collection of k-dimensional vector spaces, but now parametrized by N rather than M. Intuitively, we ought to be able to think of this collection as being "smoothly parametrized", since the map F is smooth and the set E is a "smoothly parametrized" collection of vector spaces. In other words, the set

$$\coprod_{p\in N} E_{f(p)} \tag{2.1}$$

ought to carry a natural vector-bundle structure (with base-space N), induced by the smooth map f and the bundle structure of E.

This intuition is correct. The resulting vector bundle over N is called the *pullback of E by f*, denoted f^*E.

Remark 2.1 When we write "$E = \coprod_{p\in M} E_p$", the disjoint-union symbol is just a reminder that the fibers E_p are mutually disjoint; $\coprod_{p\in M} E_p = \bigcup_{p\in M} E_p$. But since a general map $f : N \to M$ need not be one-to-one, the disjoint-union symbol in "$\coprod_{p\in N} E_{f(p)}$" has a different meaning: rather than asserting that $E_{f(p_1)} \cap E_{f(p_2)} = \emptyset$ (a false assertion if there are distinct points $p_1, p_2 \in N$ such that $f(p_1) = f(p_2)$), the notation means that for a given $q \in M$, we are associating a separate copy of E_q to each $p \in f^{-1}(q)$, and retaining the label p for the copy that arose from p.

This labeling is equivalent to the statement that, as a set, "$\coprod_{p\in N} E_{f(p)}$" iimplicitly means either $\coprod_{p\in N}(\{p\}\times E_{f(p)})$, a union of pairwise-disjoint of subsets of $N\times E$, or $\coprod_{p\in N}(E_{f(p)}\times\{p\})$, a union of pairwise-disjoint of subsets of $E\times N$. To do the concrete constructions of f^*E below, we have to choose either the "$N\times E$" meaning of the labeling or the "$E\times N$" meaning. It is customary to use the "$N\times E$" meaning, as we do in the constructions below.

The bundle f^*E can be defined in several equivalent ways. More precisely, there are several constructions of "models" for f^*E that are not all identical (e.g. some may be different as point-sets) but are all canonically isomorphic to each other. Although the models are equivalent, some directly yield insights into a particular concept that are less apparent with other models.

We give two closely-related constructions of f^*E (actually the same construction from two different points of view) below. The first construction is faster, while the second gives some geometric insight into *pulled-back connections*, the topic of Section 3.

Below, we assume we have been given M and N, a rank-k vector bundle $E\stackrel{\pi}{\to}M$, and a smooth map $f:N\to M$.

Construction 1 of f^*E.

First observe that there is a 1-1 correspondence

$$\begin{aligned}\textstyle\coprod_{p\in N} E_{f(p)} &\longleftrightarrow \textstyle\coprod_{p\in N}\left(\{p\}\times E_{f(p)}\right)\subset N\times E &(2.2)\\ &= \{(p,v)\in N\times E : f(p)=\pi(v)\}. &(2.3)\end{aligned}$$

We take the RHS of (2.3) to be the definition of f^*E as a *set*. It is not hard to show that f^*E is a submanifold of $N\times E$, of dimension $\dim(N)+k$.

Let $\mathrm{proj}_1 : N\times E\to N$ and $\mathrm{proj}_2 : N\times E\to E$ denote the projections onto the first and second factors, respectively, of the Cartesian product. Define maps $\pi' : f^*E\to N$ and $\tilde f : f^*E\to E$ by $\pi'=\mathrm{proj}_1|_{f^*E}$ and $\tilde f=\mathrm{proj}_2|_{f^*E}$.

Observe that, for $p\in N$,

$$\begin{aligned}(f^*E)_p &:= (\pi')^{-1}(p)\\ &= \{p\}\times\{v\in E : f(p)=\pi(v)\}\\ &= \{p\}\times E_{f(p)},\end{aligned}$$

so $\tilde f|_{(f^*E)_p}$ is a bijection $(f^*E)_p\to E_{f(p)}$. The vector-space structure on $E_{f(p)}$, together with the bijection $\tilde f|_{(f^*E)_p} : (f^*E)_p\to E_{f(p)}$, canonically induces a vector-space structure on $E_{f(p)}$. The map $\tilde f|_{(f^*E)_p} : (f^*E)_p\to E_{f(p)}$ then becomes an isomorphism.

Given any vector-bundle atlas $\mathcal{V} := \{(V_\alpha,\psi_\alpha)\}_{\alpha\in A}$ for E, let $V'_\alpha = f^{-1}(V_\alpha)$, and for $p\in V'_\alpha$ and $v'\in(\pi')^{-1}(p)$ define $\psi'_\alpha(v') = (p,\psi_{\alpha,f(p)}(\tilde f(v')))$. (Recall that for

$q \in V$, the map $\psi_{\alpha,q} : E_q \to \mathbf{R}^k$ is defined by $\psi_\alpha(v) = (q, \psi_{\alpha,q}(v))$.) It is straightforward to check that $\mathcal{V}' = \{(V'_\alpha, \psi'_\alpha)\}_{\alpha\in A}$ is a vector-bundle atlas for f^*E, and hence that f^*E is, indeed, a vector bundle over N with projection-map π'. Moreover, the map $\tilde{f} : \tilde{f}^*E \to E$ is a bundle homomorphism covering f, and restricts to an isomorphism $(f^*E)_p \to E_{f(p)}$ for each $p \in N$. We will denote the inverse of the latter isomorphism as

$$\tilde{f}_p^\sharp : E_{f(p)} \to (f^*E)_p \ . \tag{2.4}$$

(See Figure 2.)

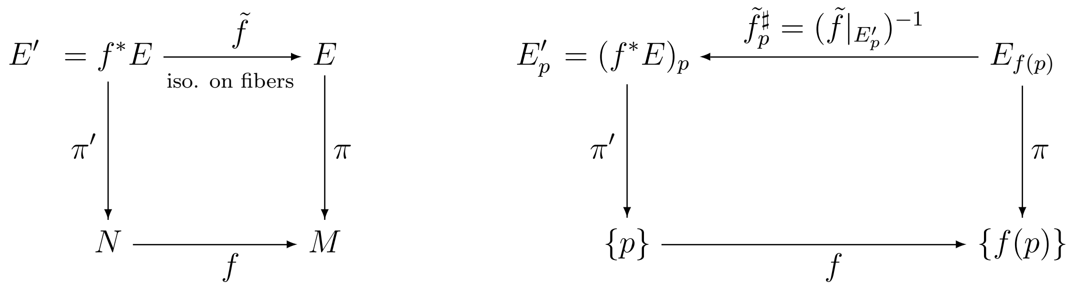

Figure 2: *Pulled-back vector bundle.* EAch of the above diagrams commutes. The pulled-back vector bundle $E' = f^*E$ comes equipped with a bundle homomorphism $\tilde{f} : E' \to E$ that, for each $p \in N$, restricts to an isomorpism $E'_p \to E_{f(p)}$.

[End of Construction 1.]

Remark 2.2 When $f : N \to M$ is a *diffeomorphism*, there is a model of f^*E that is simpler than the one above. Given such a diffeomorphism, if we define $E' = E$ and $\pi' := f^{-1} \circ \pi$, then E' is a vector bundle over N, whose fiber at $p \in N$ is $\pi'^{-1}(p) = (\pi^{-1} \circ f)(p) = \pi^{-1}(f(p)) = E_{f(p)}$. More generally, for any set $U \subset N$, we have $\pi'^{-1}(U) = \pi^{-1}(f(U))$, so for any $V \subset N$ we have $\pi'^{-1}(f^{-1}(V)) = \pi^{-1}(V)$. Hence if $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ is a vector-bundle atlas for E, then $\{(f^{-1}(V_\alpha), \psi_\alpha)\}$ is a vector-bundle atlas for E'. This bundle E' is canonically isomorphic to f^*E: if we define $H : E' = E \to f^*E$ by

$$H(v) = (\pi(v), v) \in \{\pi(v)\} \times E'_p = \{\pi(v)\} \times E_{f(\pi(v))} = (f^*E)_{\pi(v)} \,,$$

then H is a bundle homomorphism covering id_N and restricting to an isomorphism on each fiber. However, the underlying point-set of E' is *literally* $\coprod_{p\in N} E_{f(p)}$ rather than the set $\coprod_{p\in N} \{p\} \times E_{f(p)}$ that is in natural 1-1 correspondence with $\coprod_{p\in N} E_{f(p)}$ (cf. (2.2)).

<u>Construction 2.</u> Let $\tilde{E} = N \times E$, define $\tilde{\pi} : \tilde{E} = N \times E \to N \times M$ by $\tilde{\pi}(p, v) = (p, \pi(v))$; i.e. $\tilde{\pi} = \mathrm{id}_N \times \pi$.

We claim that $\tilde{E} \xrightarrow{\tilde{\pi}} N \times M$ "is" a rank-k vector bundle; i.e. that naturally carries the structure of a rank-k vector bundle. To establish this, again define $\text{proj}_2 : N \times E \to E$ by $\text{proj}_2(p, v) = v$. Then

$$\tilde{E}_{(p,q)} := \tilde{\pi}^{-1}(p,q) = \{p\} \times E_q \tag{2.5}$$

for all $(p,q) \in N \times M$. For each $p \in N$, the map $\text{proj}_2|_{\{p\}\times E_q} : \{p\} \times E_q = \tilde{E}_{(p,q)} \to E_q$ is a bijection. This bijection, combined with the vector-space structure on E_q, defines a vector-space structure on $\tilde{E}_{(p,q)}$, making $\text{proj}_2|_{\tilde{E}_{(p,q)}} : \tilde{E}_{(p,q)} \to E_q$ an isomorphism.

Next, let $\overline{\text{proj}_1} : N \times M \to N$ denote the map $(p,q) \mapsto p$. For any set $V \subset M$, observe that $\tilde{\pi}^{-1}(N \times V) = N \times \pi^{-1}(V)$. Given a vector-bundle chart (V, ψ) of E, let $V' = N \times V$ and define

$$\psi' : \tilde{\pi}^{-1}(N \times V) = N \times \pi^{-1}(V) \to N \times (V \times \mathbf{R}^k) = (N \times V) \times \mathbf{R}^k$$

by $\psi' = \text{id}_N \times \psi : N \times \pi^{-1}(V) \to (N \times V) \times \mathbf{R}^k$. It is easily seen that (V', ψ') is a vector-bundle chart for $\tilde{E}$. Applying the same procedure to each chart in a vector-bundle atlas $\mathcal{V} := \{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$ for E, we obtain a collection of vector-bundle charts $\mathcal{V}' := \{(V'_\alpha, \psi'_\alpha)\}_{\alpha \in A}$ for E', and the smooth-overlap condition for $\mathcal{V}'$ is easily checked. Hence $\mathcal{V}'$ is a vector-bundle atlas for $\tilde{E}$, and $\tilde{E}$ is a vector bundle over $N \times M$.

Now define $F : N \to N \times M$ by $F(p) = (p, f(p))$. The image of F is precisely the graph of f, a submanifold of $N \times M$. Note that F, viewed as a map $\hat{F} : N \to F(N)$, is a diffeomorphism; its inverse is $\text{proj}_1|_{F(N)}$. As is easily seen, the restriction of any vector bundle to a submanifold Z of the base-space is a vector bundle over Z. In particular, $\tilde{E}|_{F(N)}$ is a vector bundle over $F(N)$, with projection $\tilde{\pi}$ (restricted to $\tilde{E}|_{F(N)}$).

Since $\hat{F}$ is a diffeomorphism, we can identify $\tilde{E}|_{F(N)}$ with a vector bundle over N whose projection-map is $\pi' = \text{proj}_1 \circ \tilde{\pi} : \tilde{E}|_{F(N)} \to N$, as in Remark 2.2. (See Figure 3.)

For each $p \in N$,

$$\begin{aligned}
(f^*E)_p &= (\tilde{\pi}|_{\tilde{E}|_{F(N)}})^{-1}\left((\text{proj}_1|_{F(N)})^{-1}(p)\right) \\
&= \tilde{\pi}^{-1}(F(p)) \\
&= \tilde{\pi}^{-1}(p, f(p)) \\
&= \{p\} \times E_{f(p)} \qquad \text{(recall (2.5))}.
\end{aligned}$$

Thus, as in Construction 1, the fiber $(f^*E)_p$ is canonically (and isomorphically) identified with $E_{f(p)}$.

[End of Construction 2.]

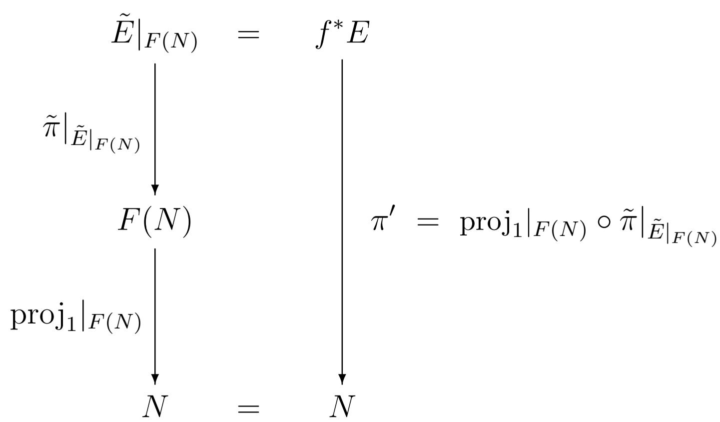

Figure 3: Second construction of $f^*E \xrightarrow{\pi'} N$.

Remark 2.3 As seen in both constructions, there is a natural bundle map $\tilde{f} : f^*E \to E$. But for general maps $f : N \to M$, there is no natural bundle map from E to f^*E. However, on the level of sections, there *is* a natural linear map $f^\sharp : \Gamma(E) \to \Gamma(f^*E)$ (a pullback map on sections) defined by

$$(f^\sharp s)(p) = f_p^\sharp(s(f(p))) \tag{2.6}$$

(see (2.4)). It often convenient to leave $f_p^\sharp$ implicit in expressions like the RHS of (2.6), and to implicitly identify $(f^*E)_p$ with $E_{f(p)}$. If we do this, then the pullback-equation (2.6) becomes simply $f^\sharp s = s \circ f$, the familiar formula for pullback of functions.

3 Pulled-back connections

Assume we have been given manifolds M and N, a rank-k vector bundle $E \xrightarrow{\pi} M$, and a smooth map $f : N \to M$. Let $\tilde{E} \xrightarrow{\tilde{\pi}} N \times M$ and other notation be as in "Construction 2" in the previous section. Each $p \in N$ determines an inclusion map $j_p : M \hookrightarrow N \times M$ with image $\{p\} \times M$ (the map $q \mapsto (p,q)$). Similarly, each $q \in M$ determines an inclusion map $\iota_q : N \hookrightarrow N \times M$ with image $N \times \{q\}$ (the map $p \mapsto (p,q)$). Specifically, these maps are defined by

$$j_p(q) = (p,q) = \iota_q(p) \quad \text{for all } p \in N,\ q \in M.$$

A connection ∇ on E naturally determines a connection $\widetilde{\nabla}$ on $\tilde{E}$, as follows. Let $s \in \Gamma(\tilde{E})$ (the space of sections of $\tilde{E}$). For each $p \in N$, define a section $j_p^* s \in \Gamma(E)$ by $j_p^* s = \mathrm{proj}_2 \circ s \circ j_p$. More visually, $j_p^* s : M \to E$ is the map

$$q \in M \longmapsto s(p,q) \in \tilde{E}_{(p,q)} = \{p\} \times E_q \longmapsto \mathrm{proj}_2(s(p,q)) \in E_q\ .$$

For each $q \in M$, define a function $\iota_q^* s : N \to E_q$ by $\iota_q^* s = \mathrm{proj}_2 \circ s \circ \iota_q$. More visually, $\iota_q^* s$ is the map

$$p \in N \longmapsto s(p,q) \in \tilde{E}_{(p,q)} = \{p\} \times E_q \longmapsto \mathrm{proj}_2(s(p,q)) \in E_q \; .$$

Note that for each $p \in N$, $q \in M$, the function $\iota_q^* s$ takes its values in the *fixed*—k-dimensional vector space E_q. For each while for $p \in N$, the object $j_p^* s$ is a more complicated object: a secton of the vector bundle E.

For $(p,q)\iota N \times M$, the tangent space $T_{(p,q)}(N \times M)$ may be canoniically identified with $T_pN \oplus T_qM$. We use this to write a general element of $T_{(p,q)}(N \times M)$ as (X_p, Y_q), where $X_p \in T_pN$ and $Y_q \in T_qM$. For $(X_p, Y_q) \in T_{(p,q)}(N \times M)$, define

$$\tilde{\tilde{\nabla}}_{(X_p,Y_q)} s := X_p(\iota_q^* s) + \nabla_{Y_q}\, j_p^* s \tag{3.7}$$

In equation (3.7), "$X_p(\iota_q s)$" denotes the ordinary directiional-derivative, in the direction X_p at $p \in N$, of the function $\iota_q^s : N \to E_q$, an ordinary vector-valued function on N (taking values in the *siingle* vector space E_q). Note tjat both summands on the RHS of (3.7) lie in the vector space $E_q = \tilde{E}_{(p,q)}$, so $\tilde{\tilde{\nabla}}_{(X_p,Y_q)} s$ lies in $\tilde{E}_{(p,q)}$ as well.

Exercise 3.1 Check that equation (3.7) defines a connection $\tilde{\tilde{\nabla}}$ on $\tilde{E}$.

Now consider a section $s \in \Gamma(f^*E)$. Using the diffeomorphism $\overline{\mathrm{proj}_1}|_{F(N)} : F(N) \to N$, we may identify f^*E with the bundle $\tilde{E}|_{F(N)}$ as ain the last step of "Construction 2" of f^*E. Siince $F(N)$ is a submanifold of $N \times M$, the section s may be differentiated using $\tilde{\tilde{\nabla}}$: at any point of $(p, f(p)) \in F(N)$, we extend s locally to a section of $\tilde{E}|_U$ on some open neighborhood U of $(p, f(p)$, and use $\tilde{\tilde{\nabla}}$ to covariantly differentiate $\tilde{s}$ in directions tangent fo $F(N)$; the result is independent of the choice of extension. (*Exercise*: check this "independent of choice of extension" property for a connection on a general vector bundle $E' \to M'$ and a submanifold $Z \subset M'$.)

Thus, for $p \in N$ and $X_p \in T_pN$, we can unambiguously define

$$\begin{aligned}
(f^*\nabla)_{X_p} s \;&:=\; \tilde{\tilde{\nabla}}_{F_{*p}X_p}\, s \\
&:=\; \tilde{\tilde{\nabla}}_{(X_p, f_{*p}X_p)} \tilde{s} \quad \text{(where } \tilde{s} \text{ is any local extension of } s \\
&\qquad\qquad\qquad \text{to a nbhd of } (p, f(p)) \text{ in } N \times M) \\
&:=\; X_p(\iota_{f(p)}^* \tilde{s}) + \nabla_{f_{*p}X_p}(j_p^* \tilde{s}) \quad \text{(by (3.7))}.
\end{aligned}$$

Allowing p to vary, we then obtain a map $f^*\nabla : \Gamma(TN) \times \Gamma(f^*E) \to \Gamma(f^*E)$.

Exercise 3.2 Check that $f^*\nabla$ is a connection on f^*E.

Definition 3.3 The connection $f^*\nabla$ on f^*E is the *pullback*, by f, of the connection ∇ on E.