Differential Geometry—MTG 6257—Spring 2026
Problem Set 1
Due-date: Wednesday 2/11/26)

Required problems (to be handed in): 1a, 2, 3e(i). In doing any of these problems, you may assume the results of all earlier problems (optional or required).

Optional problems: All the ones that are not required.

Required reading: All the problems.

Recommended reading: the notes "Bump-functions and the locality of Leibnizian linear operators" (linked to the class home page). This material was covered in class, but the presentation is cleaner in these notes than it was in class.

1. **Extensions from a closed submanifold.** This problem is another valuable application of partitions of unity. You should find the arguments for all three parts very similar to each other.

Let M be a manifold, $Z \subset M$ a submanifold that is closed as a subset of M.

(a) "*Smooth Tietze Extension Theorem*". Suppose $f : Z \to \mathbf{R}$ is a smooth function. Show that f can be extended to a smooth function $M \to \mathbf{R}$.

Note: This would be false without the hypothesis that Z is closed in M, even if we were looking just for *continuous* extensions, and even if we required $\dim(Z)$ to be strictly smaller than $\dim(M)$. (Example: $M = S^2$, $Z =$ equator $\setminus$ {one point}.) If your argument doesn't use the hypothesis that Z is closed, you've made a mistake. The same goes for parts (b) and (c).

(b) A *vector field along* Z is a section of $TM|_Z$, i.e. a smooth map $X : Z \to TM$, $p \mapsto X_p \in T_pM$. (We do not require X_p to be tangent to Z.) Show that a vector field along Z can be extended to a vector field on M.

(c) Similarly, for $k > 0$ a *k-form along* Z is a map $\omega : Z \to \bigwedge^k T^*M$, $p \mapsto \omega_p \in \bigwedge^k T_p^*M$, smooth in the sense that if $X_1, \ldots X_k$ are locally defined vector fields along Z (smooth by definition), then the real-valued function $\omega(X_1, \ldots, X_k)$ [i.e. the function $p \mapsto \omega(X_1, \ldots, X_k)\big|_p$] is smooth.[1] Show that a k-form along Z can be extended to a k-form on M.

[1]Here, "locally" means "defined on a Z-open subset of Z." However, on any manifold N, for each $q \in N$ we can define the space $\mathcal{G}_q(N; TN)$ of *germs at q of vector fields on N* analogously to the way we defined germs at q of smooth real-valued functions. For a fixed k-form ω, smoothness at p of a map of the form $\omega(X_1, \ldots, X_k)$ in the problem, where the X_i are defined on (potentially) just some open neighborhood of p, clearly depends only on the *germs* of the X_i at p. An argument similar to the proof of Proposition 1.4 in the notes "Bump-functions and the locality of Leibnizian linear operators" shows that every germ of a vector field on N has a representative that is a vector field defined on all of N. The upshot is that, in the definition of smoothness of ω, we can remove "locally

2. Prove the *"Smooth Urysohn Lemma"*: Let M be a manifold, let $Z \subset M$ be a closed set (not necessarily a submanifold), and let $U \subset M$ be an open neighborhood of Z. Then there exists a smooth function $\tilde{\chi} = \tilde{\chi}_{Z,U} : M \to [0,1]$, such that $\tilde{\chi}$ is identically 1 on Z and is identically 0 on $M \setminus U$.

3. (Parts (a)–(c) of this problem were assigned informally at the blackboard last semester, but did not appear in problem-sets. For students who did all the "blackboard assignments", these parts will be a review.)

Let M, N, Z be manifolds. Recall that given a smooth map $F : M \to N$ and a k-form ω on N, with $k > 0$, the *pullback of* ω *by* F is the k-form $F^*\omega$ on M defined by

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(F_{*p}\, v_1, \ldots, F_{*p}\, v_k) \quad \forall p \in M \text{ and } v_1, \ldots, v_k \in T_pM. \quad (1.1)$$

(a) Let $F : M \to N$ be a smooth map and let $k \geq 0$. Show that the map $\Omega^k(N) \to \Omega^k(M)$ given by $\omega \mapsto F^*\omega$ is linear.

(b) Let $F : M \to N$ be a smooth map and let ω, η be differential forms on N. Show that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$. (Do not forget the case in which the degree of ω or η is zero.)

(c) Let $F : M \to N$ and $G : N \to Z$ be smooth maps, and let ω be a differential form (of arbitrary degree) on Z. Show that $(G \circ F)^*\omega = F^*(G^*\omega)$.

(d) Show that if $F : M \to N$ is a diffeomorphism, then the linear map $F^* : \Omega^k(N) \to \Omega^k(M)$ is invertible for each k, with inverse given by $(F^*)^{-1}\eta = (F^{-1})^*\eta$.

(e) Recall that although individual tangent *vectors* push forward under any smooth map of manifolds, vector *fields* neither push forward nor pull back under arbitrary maps. However, under a diffeomorphism, both push-forwards and pullbacks of vector fields are well-defined, as follows:

(i) Suppose that $F : M \to N$ is a diffeomorphism and X is a vector field on M. For each $q \in N$ we define

$$(F_*X)_q \;=\; F_{*F^{-1}(q)}\big(X_{F^{-1}(q)}\big) \in T_qN.$$

Thus the map $F_*X : N \to TN$ defined by the assignment $q \mapsto (F_*X)_q$ is a "set-theoretic" vector field (an assignment of an element of T_qN to each $q \in N$, without regard to smoothness or even continuity). Show that F_*X is smooth, hence is a (true) vector field on N.

(ii) Suppose that $F : M \to N$ is a diffeomorphism and Y is a vector field on N. Then $F^{-1} : N \to M$ is also a diffeomorphism, so by part (i) we can define a vector

defined" from "locally defined vector fields" without changing which ω's we're calling smooth.

field F^*Y on M by

$$F^*Y \;=\; (F^{-1})_*Y.$$

(f) Show that if $F : M \to N$ is a diffeomorphism, $\omega \in \Omega^1(N)$, and X is a vector field on N, then

$$F^*(\langle \omega, X\rangle) = \langle F^*\omega, F^*X\rangle. \tag{1.2}$$

(In (1.2), $\langle \omega, X\rangle$ is the function $p \mapsto \langle \omega_p, X_p\rangle = \omega_p(X_p)$; both sides of the equation are real-valued functions on M.)

4. ("Raising and lowering indices"). (Some parts of this problem were done in class last semester and/or earlier this semester. Most or all of the remaining parts of this problem were assigned informally at the blackboard but did not appear in problem-sets. For students who did all the "blackboard assignments", these parts will be a review.)

Let V be a finite-dimensional vector space and let g be an inner product on V. Let $\mathsf{g} : V \to V^*$ be the map $v \mapsto g(v, \cdot)$.

(a) Show that $\mathsf{g} : V \to V^*$ is an isomorphism.

For reasons seen in part (e) below, we informally refer to g and g^{-1} its inverse as "index-lowering" or "index-raising" isomorphisms.

(b) Define a function $\tilde{g} : V^* \times V^* \to \mathbf{R}$ by $\tilde{g}(\alpha, \beta) = g\big(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\beta)\big)$. Show that $\tilde{g}$ is an inner product on V.

(c) Show that, for all $\alpha \in V^*$ and $v \in V$,

$$\tilde{g}\big(\alpha, \mathsf{g}(v)\big) = \langle \alpha, v\rangle = g\big(\mathsf{g}^{-1}(\alpha), v\big). \tag{1.3}$$

(Here $\langle \cdot, \cdot\rangle$ is the dual pairing $V^* \times V \to \mathbf{R}$; $\langle \alpha, v\rangle = \alpha(v)$.)

For the remainder of this problem, let $\{e_i\}_{i=1}^n$ be a basis of V, let $\{\theta^i\}_{i=1}^n$ be the dual basis of V^*, and for all $i, j \in \{1, \dots, n\}$ define $g_{ij} = g(e_i, e_j)$ and $g^{ij} = \tilde{g}(\theta^i, \theta^j)$. Let $g_{\cdot\cdot}$ and $g^{\cdot\cdot}$ denote the $n \times n$ matrices whose $(ij)^{\text{th}}$ entries are g_{ij} and g^{ij} respectively. Below we will use *Einstein summation convention*, "sum over repeated indices."[2]

[2]More precisely: in any expression written as a product of one or more factors labeled by indices, with the same index repeated in two factors, there is an implicit sum over that index. (E.g. $c_{ij}\, d^j := \sum_j c_{ij}\, d^j$; $c_{ij}\, d_k{}^j := \sum_j c_{ij}\, d^j{}_k$; $A^i{}_i := \sum_i A^i{}_i$; and $a_{ij}\, b^{jk}\, c_{ki} := \sum_{i,j,k} a_{ij}\, b^{jk}\, c_{ki}$. (The summation ranges are determined by context, and need not be the same for different indices.) This convention is generally applied only in the context of linear or multilinear algebra (tensor algebra); the labeled objects are generally either numbers or vectors. In this context, if proper "index hygiene" has been used, a repeated index will usually occur once as an upper index and once as a lower index. To avoid confusion, it is best to avoid summation convention for any index that appears more than twice in the same factor or product (e.g. it's best not to omit the explicit "$\sum_i$" in "$\sum_i c^i \lambda_i v_i$").

(d) Show that $g_{\bullet\bullet}$ is the matrix of $\mathtt{g} : V \to V^*$ with respect to the given bases, and that $g^{\bullet\bullet}$ is the matrix of $\mathtt{g} : V \to V^*$ is the matrix of $\mathtt{g}^{-1} : V^* \to V$ with respect to these bases.[3] It follows that $g^{\bullet\bullet} = (g_{\bullet\bullet})^{-1}$.

(e) For $1 \leq i \leq n$ define $\theta_i = \mathtt{g}(e_i)$ and $e^i = \mathtt{g}^{-1}(\theta^i)$. (Do not confuse θ_i with θ^i, or e^i with e_i, but the notation has been chosen so that all the θ_i and θ^i live in V^*, and all the e_i and e^i live in V.)

Let $v, w \in V$, let $\alpha, \beta \in V^*$, let $\{a^i\}$ and $\{b^i\}$ be the coordinates of v and w with respect to the basis $\{e_i\}$, and let $\{c_i\}$ and $\{d_i\}$ be the coordinates of v and w with respect to the basis $\{\theta^i\}$. (Thus $v = a^i\, e_i,\ \ \alpha = c_j\, \theta^j$, etc.[4])

For $1 \leq i \leq n$ define $a_i = g_{ij}\, a^j$ and $b_i = g_{ij}\, b^j$ (using multiplication by the $g_{\bullet\bullet}$—the matrix of $\mathtt{g}$ with respect to the given bases—to "lower an index"); define $c^i = g^{ij}\, c_j$ and $d^i = g^{ij}\, d_j$ (using multiplication by $g^{\bullet\bullet}$—the matrix of $\mathtt{g}^{-1}$ with respect to the given bases— to "raise an index").

Check that the following equalities hold. (Some of these equalities just repeat definitions; I've included them for extra visual emphasis of a "conservation of indices" principle.[5]) *Note*: the list of equalities continues on the next page.

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[3]By definition of "matrix of a linear transformation with respect to bases in the domain and codomain", the problem is asking you to show that $\mathtt{g}(e_j) = \theta^i g_{ij}$ and that $\mathtt{g}^{-1}(\theta_j) = e_i g^{ij}$, $1 \leq j \leq n$, where we are allowing ourselves to write the product of a vector and a scalar with the factors in either order.

You may recall my saying previously that for a linear transformation $T : X \to Y$, where X and Y are finite-dimensional vector spaces with bases $\{v_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^m$ respectively, then for the matrix A of T with respect to these bases, good "index hygiene" dictates writing the $(ij)^{\text{th}}$ entry as ${A^i}_j$. However, the upper/lower positioning of indices in this matrix arises from, and is opposite to, the upper/lower nature of the index used to label the corresponding bases. If $\{\phi^i\}_{i=1}^n$is the basis of X^* dual to $\{v_i\}$, then $T = {A^i}_j\, \phi^j \otimes w_i$ under the canonical identification of $\mathrm{Hom}(X, Y)$ with $X^* \otimes Y$ (a canonical isomorphism we discussed last semester). However, if $Y = X^*$ then, given the basis $\{v_i\}$ of X, it is natural to choose the dual basis $\{\phi^i\}$ as the basis for Y. Labeling these basis-elements with an upper index leads us to write the corresponding index of A as a lower index: $T = A_{ij} \phi^j \otimes \phi^i$. The first index of a matrix-entry still labels the row, and the second labels the column, regardless of upper/lower positioning, so we don't stack an upper index on top of a lower index (with one exception: we allow ourselves the laziness of writing δ^i_j instead of ${\delta^i}_j$ or ${\delta_j}^i$).

[4]In good "index hygiene", given *general* bases (with no extra properties, such as orthonormality with respect to a given inner product), the upper/lower position of the coordinate-indices is opposite to the positioning of the basis-indices. This reflects the fact that coordinate *functions*, with respect to a basis of a vector space W, are elements of the dual space W^*.

[5]This principle is that any "free" index—one that is not summed over—appears with the same upper/lower positioning on both sides of each equation. (The term "conservation of indices" is my own informal name for this.) Summed-over indices are dummy indices, having no meaning outside the summation, so they don't need to appear on both sides of an equation; e.g. $a^i e_i = a^j e_j$.

$$\begin{aligned} \mathsf{g}(e_i) &= \theta_i &= g_{ij}\theta^j \ , \quad 1 \le i \le n. \\ \mathsf{g}^{-1}(\theta^i) &= e^i &= g^{ij}e_j \ , \quad 1 \le i \le n. \end{aligned}$$

$$g(e^i, e_j) = \delta^i_j = \tilde{g}(\theta^i, \theta_j), \qquad i, j \in \{1, \ldots, n\}.$$

$$\begin{aligned} \langle \alpha, v \rangle &= b^i\, c_i &= b_i c^{\,i} \ . \\ \mathsf{g}(v) = \mathsf{g}(b^i\, e_i) &= b^i\, \theta_i &= b_i\, \theta^i. \\ \mathsf{g}^{-1}(\alpha) = \mathsf{g}^{-1}(c_i\, \theta^i) &= c_i\, e^i &= c^i\, e_i \ . \\ g(v, w) = g_{ij}\, a^i\, b^j &= a^j\, b_j &= a_i\, b^i \ . \\ \tilde{g}(\alpha, \beta) = g^{ij}\, c_i\, d_j &= c^j\, d_j &= c_i\, d^i \ . \end{aligned}$$

(f) Rewrite equation (1.3) using the notation of part (e).

(g) Show that if the basis $\{e_i\}$ is orthonormal, then so is the basis θ^i, and furthermore $\theta_i = \theta^i$ and $e^i = e_i$ (in the notation of part (e)).

Thus, for equations involving bases that are special for some *extra structure*—e.g. an inner product—that's been chosen for some vector space, "conservation of indices" does not always hold.