Differential Geometry—MTG 6257—Spring 2026
Problem Set 2
Due-date: Friday 3/6/26

Required *reading*: all the problems.

Required problems (to be handed in): 1e, 2h(ii), 6, 7, 8abc. In doing any of these problems, you may assume the results of all earlier problems (optional or required). Problems 1–5 were designed to be done *in the given order, without skipping anything*, although problem 3 and problem-parts 1bd don't help with later problems. You should not even *start* on any hand-in part of problem 2 without first reading-through problem 1 (at least parts a,c,e) and all prior parts of problem 2.

Optional problems: All the ones that are not required.

1. Let $n \geq 1$, let M be an n-dimensional oriented manifold, and let $D \subset M$ be a domain with regular boundary.

(a) Show that $M \setminus D$ is also a domain with regular boundary. (As sets, $\partial(M \setminus D) = \partial D$, of course.)

(b) Since ∂D is the boundary of each of the two domains-with-regular-boundary D and $M \setminus D$ the manifold ∂D inherits one orientation as the boundary of D and another as the boundary of the $M \setminus D$. How do these orientations compare?

For the remainder of this problem, fix $\omega \in \Omega_c^n(M)$.

(c) Since both D and $M \setminus D$ are domains with regular boundary, both $\int_D \omega$ and $\int_{M \setminus D} \omega$ are defined. Show that

$$\int_M \omega = \int_D \omega + \int_{M \setminus D} \omega. \tag{1.1}$$

(d) Suppose $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(M)$. Observe that Stokes's Theorem can be applied to each of the three integrals in (1.1). Check that your answer to part (c) is consistent with equation (1.1).

(e) Let $p \in M$. For $\epsilon > 0$, let $B_\epsilon(\vec{0})$ and $\bar{B}_\epsilon(\vec{0})$ denote, respectively, the open and closed Euclidean ball of radius ϵ centered at $\vec{0} \in \mathbf{R}^n$. Let (U, ϕ) be a chart of M "centered" at p—i.e. a chart with $p \in U$ and for which $\phi(p) = \vec{0}$. Then $\bar{B}_\epsilon(\vec{0}) \subset U$ for all sufficiently small $\epsilon > 0$. Restricting attention to such $\epsilon > 0$ henceforth, let $D_\epsilon = \phi^{-1}(B_\epsilon(\vec{0}))$. It is easily seen that D_ϵ is a domain with regular boundary (check this, but do not hand it in). Show that $\lim_{\epsilon \to 0} \int_{D_\epsilon} \omega = 0$, and deduce that

$$\int_M \omega = \lim_{\epsilon \to 0} \int_{M \setminus D_\epsilon} \omega. \tag{1.2}$$

2. **(Measure-zero sets in $\mathbf{R}^n$ and manifolds).**

For $\lambda > 0$, let us call any translate of $[0,\lambda]^n = [0,\lambda] \times [0,\lambda] \times \cdots \times [0,\lambda]$ in $\mathbf{R}^n$ (where $n \in \mathbf{N}$ is given) an *n-cube of side λ*. (Equivalently, an n-cube of side λ is a closed ℓ^∞-ball of radius $\lambda/2$ in $\mathbf{R}^n$.) For any n-cube C of side λ, we define the *n-measure* (or *n-volume*) *of C* to be $\mu_n(C) := \lambda^n$. When the intended n is clear from context, we may simply write $\mu(C)$ and call this the *measure* of C.

> **Definition (measure-zero subset of $\mathbf{R}^n$).** A set $Z \subset \mathbf{R}^n$ has *(n-)measure zero* if, for all $\epsilon > 0$, Z can be covered by a countable[1] collection of cubes $\{C_i\}$ for which $\sum_i \mu_n(C_i) < \epsilon$.

(The definition above coincides with the definition of "Lebesgue-measure-zero subset of $\mathbf{R}^n$". However, we are *not* doing measure theory here. We are not defining "measurable set", or "measure-(anything other than zero) subset of $\mathbf{R}^n$." The notion of "measure zero" requires no measure theory; it is a much more primitive concept. For the integration of (smooth, or even just continuous) differential forms over manifolds, or domains with regular boundary, the *Riemann* integral[2] suffices, both in definition and proofs.)

Clearly, if $Z \subset \mathbf{R}^n$ has measure zero, then so does any subset of Z.

(a) Show that if, in the definition of "measure-zero subset of $\mathbf{R}^n$", we use "open n-cubes $(0,\lambda)^n$", exactly the same subsets of $\mathbf{R}^n$ have measure zero. (Hence, at any time later in this problem, if you find it convenient to use covers by open n-cubes instead of closed n-cubes, you may do so.)

(b) Show that a *compact* subset $Z \subset \mathbf{R}^n$ has n-measure zero if and only if for all $\epsilon > 0$, Z can be covered by a *finite* collection of cubes $\{C_i\}$ for which $\sum_i \mu_n(C_i) < \epsilon$.

(c) For $k \in \mathbf{N}$ with $k < n$, show that $\mathbf{R}^k \times \{0_{\mathbf{R}^{n-k}}\}$ has n-measure zero.

(d) Show that any countable union of measure-zero subsets of $\mathbf{R}^n$ has measure zero.

(e) Let $U \subset \mathbf{R}^n$ be a nonempty open set, let $F : U \subset \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map.

(The only difference between this version of this problem-part and the original version is that "Let $K \subset U$ be compact" is now introduced individually in parts (i) and (ii), rather than K being the same compact set in both problem-parts. Thus, sub-part (i) can be applied to any compact subset $K' \subset U$, e.g. any new compact subset that may be constructed from the K that appears in the statement of part (ii).)

(i) Let $K \subset U$ be compact. Show that there exists a constant $b > 0$ such that if C is a cube of side λ contained in K, then $F(C)$ is contained in a cube of side $b\lambda$.

[1] I adhere to the convention countable sets may be finite or countably infinite. However, whether we use this convention or the one in which "countable" means "countably infinite", exactly the same subsets of $\mathbf{R}^n$ have measure zero.

[2] more precisely, the multivariable generalization that's often called the *Riemann-Jordan integral.*

(ii) Let $Z \subset U$ and let $K \subset U$ be compact. Show that if $\mu_n(Z \cap K) = 0$, then $\mu_n(F(Z \cap K)) = 0$.

One strategy: construct a compact set K', with $K \subset K' \subset U$, such that for any $\epsilon > 0$, there is a countable collection $\{C_i\}$ of cubes that lie in K', cover $Z \cap K$, and satisfy $\sum_i \mu_n(C_i) < \epsilon$. (Some care must be taken in the construction, since there can be cubes appearing in the definition of "$Z \cap K$ has measure zero" that do not even lie in U.) Then apply part (i).

(f) Let U and F be as in part (e). Recall that U admits an exhaustion by compact subsets: a nested, increasing sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \dots$ with $\bigcup_{i=1}^{\infty} K_i = U$. (One such exhaustion can be constructed as follows. Let $\{p_i\}_{i=1}^{\infty}$ be an enumeration of the points in U all of whose coordinates are rational. Choose any norm on $\mathbf{R}^n$, and for each $j \in \mathbf{N}$, let $r_j > 0$ be such that the closed ball $V_j := \bar{B}_{r_i}(p_i)$—as defined by the given norm—lies in U. Then $\bigcup_{j=1}^{\infty} V_j = U$. For each $i \in \mathbf{N}$, let $K_i = \bigcup_{j=1}^{i} V_j$. Then $(K_i)_{i=1}^{\infty}$ is a nested, increasing sequence of compact subsets of U, and $\bigcup_{i=1}^{\infty} K_i = \bigcup_{j=1}^{\infty} V_j = U$.) Use this, and earlier parts of this problem, to show that if $Z \subset U$ and $\mu(Z) = 0$, then $\mu(F(Z)) = 0$.

Thus, for any open set $U \subset \mathbf{R}^n$ **a C^1 map $F : U \to \mathbf{R}^n$ carries sets of (n-)measure zero to sets of (n-)measure zero.**

(g) Let $U, V \subset \mathbf{R}^n$ be open sets and let $F : U \to V$ be a diffeomorphism. (Recall that our convention is that "diffeomorphism" means "C^∞ diffeomorphism". But for this problem-part, the argument is the same whether we require diffeomorphisms to be C^1, C^∞, or anything in between.) Let $Z \subset U$. Show that $\mu_n(Z) = 0 \iff \mu_n(F(Z)) = 0$.

Definition (measure-zero subset of a manifold). Let M be an n-dimensional manifold with maximal atlas $\mathcal{A}$, and let $Z \subset M$. We say that Z has *measure zero* (or *measure zero in* M), and write $\mu_n(Z) = 0$ (or simply $\mu(Z) = 0$ when no confusion can arise) if for every chart $(U, \phi) \in \mathcal{A}$, $\mu_n(\phi(Z \cap U)) = 0$.

Note that since $\mathbf{R}^n$ is itself an n-dimensional manifold, we now have two potentially different meanings of "measure zero subset of $\mathbf{R}^n$. Part (h)(i), below, shows that there is no ambiguity.

(h) Let M be an n-dimensional manifold and let $Z \subset M$.

(i) Show that if M has an atlas $\mathcal{A}' = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ (within the implicitly given maximal atlas) such that $\mu_n(\phi_\alpha(Z \cap U_\alpha)) = 0$, then $\mu_n(Z) = 0$.

In other words: if Z can be covered by a collection of chart-domains U_α for which $\phi_\alpha(Z \cap U_\alpha)$ has measure zero for all α, then Z has measure 0. In particular this holds if $M = \mathbf{R}^n$ and $\mathcal{A}'$ is the one-chart atlas $\{(\mathbf{R}^n, \mathrm{id})\}$.

(ii) Suppose that Z is a submanifold of M with positive codimension. Show that $\mu_n(Z) = 0$.

3. **Topological aspects of measure-zero sets**.

Two facts you may assume to do this problem:

(1) No cube in $\mathbf{R}^n$ has n-measure zero. (2) Every manifold is a *Baire space*, meaning that the countable intersection of open, dense sets is dense.

Some topological terminology to recall: A subset A of a topological space is called *nowhere dense* if its closure $\bar{A}$ has empty interior (equivalently, if $\bar{A}$ contains no nonempty open set).

Clearly if $\bar{A}$ contains no nonempty open set, then neither does A. Hence the complement of a nowhere-dense set is dense.

(a) Show that a *closed* measure-zero subset of a manifold is nowhere dense. (Without "closed", this would be false, as the example of $\mathbf{Q} \subset \mathbf{R}$ shows.) Hence the complement of a closed, measure-zero subset of a manifold is open and dense.

(b) Suppose $Z \subset M$ has measure zero and is σ-compact (i.e. the countable union of compact sets). Show that the complement of Z is dense in M.

4. Recall that for a smooth map of manifolds, $F : M \to N$, (i) a critical *point* of F is a point $p \in M$ for which F_{*p} is not surjective; (ii) a critical *value* of F is a point $q \in N$ for which $F^{-1}(\{q\})$ contains a critical point; and (iii) a regular value of F is a point $q \in N$ that is not a critical value. (Note that any point of N that is not in image(F) is automatically a regular value.)

Last semester we proved the *Regular Value Theorem*: If $q \in N$ is a regular value of F, then $F^{-1}(\{q\})$ is a submanifold of M. Recall also that we briefly discussed another important theorem that's often used in conjunction with the Regular Value Theorem:

Theorem 1.1 (Sard's Theorem for C^∞ maps)[3] *Let M, N be manifolds and let $F : M \to N$ be smooth. Then the set of critical* <u>*values*</u> *of F has measure zero.*

You may assume Theorem 1.1. (For reasons of time, we are not proving it.)

Let M be a manifold, $f : M \to \mathbf{R}$ a smooth function. Show that, for almost every $c \in \text{range}(f)$, the set $f^{-1}(c)$ is a codimension-one submanifold of M, and that

[3] If $F : M \to N$ is only a C^k map, with $1 \le k < \infty$, then the conclusion of Sard's Theorem still holds if $\dim(M) \le \dim(N)$. But if $\dim(M) > \dim(N)$, then an additional assumption that $k > \dim(M) - \dim(N)$ is needed.

the *sub-level set* $\{p \in M : f(p) \leq c\}$ and *super-level set* $\{p \in M : f(p) \geq c\}$ are domains with regular boundary. Here, "almost every $c \in \mathrm{range}(f)$" means "every $c \in \mathrm{range}(f) \setminus$ (some measure-zero subset of $\mathbf{R}$)."

5. **Measure-zero sets and the computation of integrals on manifolds.**

Suppose M is a compact, oriented manifold of dimension n, and let $\omega \in \Omega^n(M)$. (Compactness of M has been assumed just to shorten this problem.) Assume we are given explicit formulas for ω—e.g., local-coordinate expressions for ω for each chart in some atlas of M. How do we use this information to *compute* $\int_M \omega$?

As mentioned in class, partitions of unity (POUs) are extremely useful for *defining* integrals n-forms on M, and for proving various theorems. But as a *computational* device, they're completely useless (other than, perhaps, for numerical integration using a computer); their formulas are intractible. Instead, we attempt to choose a collection of *disjoint* charts $\{(U_i, \phi_i)\}$ that cover all but a measure-zero subset of M, and integrate over $\bigcup_i U_i$ without using any POU; in practice we can often find a *single*, convenient chart (U, ϕ) that does the trick. (For example, in a two-chart "stereographic-projection atlas of S^n, each chart-domain covers all but a single point of S^n; in the "standard atlas" of $M = \mathbf{R}P^n$ [respectively, $\mathbf{C}P^n$], each chart-domain covers all of M except for a submanifold diffeomorphic to $\mathbf{R}P^{n-1}$ [respectively, $\mathbf{C}P^{n-1}$].) But in general, $\omega|_{U_i}$ will not have compact support, and the corresponding integral of $(\phi_i^{-1})^*\omega$ over $\mathbf{R}^n$ would be an improper integral. For example, suppose $M = S^n$, and $U = M \setminus \{p := \text{north pole}\}$, and $\phi : U \to \mathbf{R}^n$ is stereographic projection through p, then $\phi(U) = \mathbf{R}^n$. If $\omega|_U = f dx^1 \wedge \cdots \wedge dx^n$ in the corresponding local coordinates, then f will not have compact support unless ω vanishes on an open neighborhood of p. If we use stereographic projection through the *south* pole to define domains D_ϵ "centered" at p as in problem 1(e), then $\phi(M \setminus D_\epsilon) = \bar{B}_{1/\epsilon}(\vec{0})$, the closed ball of radius $1/\epsilon$ centered at the origin. Hence $\int_M \omega = \lim_{\epsilon \to 0} \int_{M \setminus D_\epsilon} \omega = \lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} (\phi^{-1})^*\omega = \lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} f$. [4]

The equality "$\int_M \omega = \lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} f$" in the S^n example is an illustration of the principle that "measure-zero sets don't affect integrals." We have not defined integrals of non-compactly-supported forms (for example, $\omega|_{S^n \setminus \{p\}}$ on the manifold $S^n \setminus \{p\}$), and existence of the single limit $\lim_{R\to\infty} \int_{\bar{B}_R(\vec{0})} f$ is not quite enough to show that $\int_{\mathbf{R}^n} f$ exists (for a general, continuous f)—but if we *did* show that $\int_{\mathbf{R}^n} f$ exists in the given example, then it would be reasonable to make the definition "$\int_{S^n\setminus\{p\}} \omega = \int_{\mathbf{R}^n} f$", and conclude that $\int_{S^n\setminus\{p\}} \omega = \int_M \omega$; i.e. that the value of the integral is unaffected if we delete the measure-zero set $\{p\}$ from M. Below, in a more general setting, you

[4] In this assignment, for various subsets $U \subset \mathbf{R}^n$, I've written "$\int_U f$" for the Riemann integral of a continuous function$f$ over U, which I've written in class with notations $\int_U f\, \mathrm{dvol}_{\mathrm{Euc}}$, $\int_U f\, \mathrm{dvol}_n$, and $\int_U f(x)$ "$dx_1 \dots dx^n$". Riemann integration suffices for all these integrals; no measure-theory is needed. However, if you *do* know Lebesgue integration: all the sets U in these instances are Lebesgue-measurable, and $\int_U f$ coincides with the integral of f over U with respect to Lebesgue measure.

will carry out a version of this argument in way that avoids some technicalities and is sufficient for computation of many integrals.

For the remainder of this problem let $Z \subset M$ be a closed subset of measure zero.

(a) Show that there exists a sequence $\{K_j \subset M\}_{j=1} \to \infty$, such that (i) $K_1 \supset K_2 \supset K_3 \supset \dots$, (ii) $\bigcap_{j=1}^{\infty} K_j = Z$; (iii) for each j, the set K_j is a compact domain with regular boundary, and (iv) $\lim_{j\to\infty} \int_{K_j} \omega = 0$.

Hint: Apply the "Smooth Urysohn Lemma" from Problem Set 1 to an appropriate, decreasing sequence $(U_j)_{j=1}^{\infty}$ of open neighborhoods of Z. Then, for each j, apply problem 4.

(b) Show that, for any sequence (K_j) as in (a), $\int_M \omega = \lim_{j\to\infty} \int_{M\setminus K_j} \omega$. Thus, if $M \setminus Z$ is contained in the domain of a chart (U, ϕ), and we define $f : \phi(U) \to \mathbf{R}$ by $(\phi^{-1})^*\omega = f dx_1 \wedge \cdots \wedge dx^n$ in the standard coordinates on $\mathbf{R}^n$, and define $V_j = \phi(U \setminus K_j) = \phi(M \setminus K_j) \subset \mathbf{R}^n$ for each j, then $\int_M \omega = \lim_{j\to\infty} \int_{M\setminus K_j} \omega = \int_{V_j} f$.

Thus we can compute $\int_M \omega$ as the limit of a sequence of integrals $\int_{V_j} f$ on $\mathbf{R}^n$, where $V_1 \subset V_2 \subset V_3 \dots$. If $\phi(M \setminus Z) = \mathbf{R}^n$ and $\int_{\mathbf{R}^n} f$ exists, then $\int_M \omega = \lim_{j\to\infty} \int_{V_j} f = \int_{\mathbf{R}^n} f$.

6. Let D be a domain with regular boundary in an oriented n-dimensional manifold M, where $n \geq 1$, and let ∂D have the induced orientation. Let $\omega \in \Omega^j(M)$, $\eta \in \Omega^k(M)$, where $j+k = n-1$, and assume that at least one of the sets $\mathrm{supp}(\omega) \bigcap \overline{D}$, $\mathrm{supp}(\eta) \bigcap \overline{D}$, is compact. (Note that the compact-support assumption is superfluous if we assume that M is compact or that $\overline{D}$ is compact.) Prove the "integration-by-parts" formula

$$\int_D d\omega \wedge \eta = \int_{\partial D} \omega \wedge \eta - (-1)^j \int_D \omega \wedge d\eta$$

(and check that this reduces to Calculus-1 integration-by-parts if $M = \mathbf{R}$, $D = [a, b]$ for some $a < b$, and $\deg(\omega) = \deg(\eta) = 0$.)

Remark. The case $D = M$ (equivalently, $\partial D = \emptyset$) is important all by itself.

7. *"Explicit" Poincaré Lemma for star-shaped regions.* The classical Poincaré Lemma asserts that, for all n and all $k > 0$, every closed k-form on $\mathbf{R}^n$ is exact. In this problem you will show, via an explicit formula, that the same result holds, more generally, for any star-shaped open subset of $\mathbf{R}^n$ (see below).

Recall that a set U in a vector space is *star-shaped* if there exists $p \in U$ such that for all $q \in U$, the line segment from p to q lies entirely in U.[5] Given such p, we may

[5]It is easily seen that star-shaped regions are *contractible*.

say that U is "star-shaped with respect to p".[6] In particular, $\mathbf{R}^n$ is star-shaped. In this problem we establish that if U is an open star-shaped subset of $\mathbf{R}^n$, then every closed k-form on U (with $k > 0$) is exact. (Thus the Poincaré Lemma follows as a special case.) There are many ways of showing this; the point of this problem is to give an explicit formula that produces, for each closed form $\omega \in \Omega^k(U)$, a form $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$.

It suffices to produce such a formula under the hypothesis that U is star-shaped with respect to the origin, which we henceforth assume; a more general formula can be obtained from this by applying a translation. The case $n = 0$ is trivial, so we also assume $n > 0$.

Set-up. For $t \in [0, 1]$ define $F_t : \mathbf{R}^n \to \mathbf{R}^n$ by $F_t(x) = tx$. Since U is star-shaped with respect to the origin, $F_t(U) \subset U$. Let V be the vector field $\sum_i x^i \frac{\partial}{\partial x^i}$. For $k > 0$ and $\omega \in \Omega^k(U)$, define $P(\omega) \in \Omega^{k-1}(U)$ by

$$P(\omega) = \int_0^1 t^{-1} F_t^*(\iota_V \omega)\, dt,$$

interpreted pointwise:

$$P(\omega)\big|_x = \int_0^1 t^{-1} \left(F_t^*(\iota_V \omega)\right)\big|_x\, dt. \tag{1.3}$$

Despite appearances, this integral is not improper: if we write ω as $\sum_I f_I\, dx^I$, where the sum is over increasing multi-indices of length k, then $\left(F_t^*(\iota_V \omega)\right)\big|_x = \sum_I t^k f_I(tx)\, \iota_{V_x} dx^I$, so the integrand in (1.3) is $O(t^{k-1})$ as $t \to 0$. **(End of set-up.)**

Your job: Show that if ω is closed, then $\omega = d(P(\omega))$ (and hence that ω is exact).

Remark 1. With $U = \mathbf{R}^3$, we have seen that there is a dictionary translating between "curl of a vector field" (interpreting "vector field" as in Calc 3) and "d of a 1-form", and "between divergence of a vector field" and "d of a 2-form". Given a vector field X such that $\nabla \cdot X = 0$, the map P above (with $k = 2$) provides one way to construct a vector field A such that $X = \nabla \times A$.[7]

Remark 2. As seen in class, for any connected manifold M we have $H^0_{\mathrm{DR}}(M) = \mathbf{R}$. Hence the Poincaré Lemma, generalized to star-shaped regions U as above, can

[6] The set U is *convex* if it is star-shaped with respect to each of its points. Thus convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible.

[7] In case you know the relevant physics: this construction of a vector potential is not terribly useful for E&M, since the regions in which we want to find vector potentials for the magnetic field, e.g. $\mathbf{R}^3$ with a curve removed (representing a current-carrying wire) are *never* star-shaped—they're not even simply connected, so they can't be contractible.

be written as

$$H^k_{\mathrm{DR}}(U) \cong \begin{cases} \mathbf{R} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases} \tag{1.4}$$

More generally, (1.4) holds under the much weaker assumption that U is *contractible* (you're not allowed to assume this until/unless I have you prove it!), but it is harder to write down an explicit formula analogous to "$P(\omega)$" in that generality.

8. Let M be a manifold. Below, the objects $\omega, \omega_1, \omega_2, \eta, \eta_1$, and η_2 are differential forms on M.

(a) Show that if ω and η are closed, then so is $\omega \wedge \eta$.

(b) Show that if ω is closed and η is exact, or vice-versa, then $\omega \wedge \eta$ is exact.

(c) For any closed form ω on M, let $[\omega]$ denote its cohomology class. For closed forms ω_1, ω_2 write $\omega_1 \sim \omega_2$ if $\deg(\omega_1) = \deg(\omega_2)$ and $\omega_1 - \omega_2$ is exact; equivalently, if $[\omega_1] = [\omega_2]$.

Suppose that $\omega_1, \omega_2, \eta_1, \eta_2$ are closed, and that $\omega_1 \sim \omega_2$ and $\eta_1 \sim \eta_2$. Show that $\omega_1 \wedge \eta_1 \ \sim\ \omega_2 \wedge \eta_2$.

(d) In view of part (c), for all $j, k \geq 0$, we may define a map $\cup : H^j_{\mathrm{DR}}(M) \times H^k_{\mathrm{DR}}(M) \to H^{j+k}_{\mathrm{DR}}(M)$ ("cup product") by

$$[\omega] \cup [\eta] = [\omega \wedge \eta]$$

(since (c) shows the RHS is independent of the choices of representatives of the cohomology classes on the LHS). There's one such map for each degree-pair (j, k), but for simplicity we use the same notation for all of them.

Show that (i) (each) cup-product map is bilinear, (ii)

$$\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha \tag{1.5}$$

for all $\alpha, \beta \in H^*_{DR}(M)$, and (iii)

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma) \tag{1.6}$$

for all $\alpha, \beta, \gamma \in H^*_{\mathrm{DR}}(M)$.

9. **Manifolds with boundary**. Recall that our original definition of "manifold M" is equivalent to one in which M *starts* as a topological space (rather than inheriting a topology from an atlas), and each chart-map is required to be homeomorphisms from an open set in M to an open subset of $\mathbf{R}^n$ (for some n). For simplicity in this problem, let us assume that we have defined "manifold" this latter way, and that we have required all charts of M to have the same dimension (a condition satisfied automatically if M is connected).

Notation: For $n \geq 1$, let $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times [0, \infty) \subset \mathbf{R}^n$, which we may call *closed upper half-space* of $\mathbf{R}^n$. We give $\mathbf{R}^n_+$ its induced topology as a subset of $\mathbf{R}^n$.

Note that the boundary of $\mathbf{R}^n_+$, *as a subset of* $\mathbf{R}^n$, is $\mathbf{R}^{n-1} \times \{0\} =: \partial\mathbf{R}^n_+$. We write the interior of $\mathbf{R}^n_+$ (as a subset of $\mathbf{R}^n$) as $\mathrm{Int}(\mathbf{R}^n_+) = \mathbf{R}^{n-1} \times (0, \infty)$.

Below, we always assume $n \geq 1$, and the abbreviation "mwb" stands for "manifold-with-boundary"

A closed domain-with-regular-boundary in a manifold is an example of a *manifold with boundary*, which can be defined by generalizing the definition of *manifold* as follows:

1. Define a *mwb chart* of a topological space M to be a pair (U, ϕ), where $U \subset M$ is open, ϕ is a map from $U \to \mathbf{R}^n_+$ for some n, and ϕ is a homeomorphism onto its image $\phi(U)$.

2. If $\hat{U}$ is an open subset of $\mathbf{R}^n_+$, we call a function $F : \hat{U} \to \mathbf{R}^n$ *smooth*, or C^∞, if F extends to a smooth map $\tilde{U} \to \mathbf{R}^n$ for some $\mathbf{R}^n$-open neighborhood $\tilde{U}$ of $\hat{U}$. If, in addition, F is a homeomorphism onto its image, and $F^{-1} : F(\hat{U}) \to \hat{U}$ is smooth, we call F a diffeomorphism.

3. An *n-dimensional mwb atlas* on a topological space M is a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of n-dimensional mwb charts of M with the property that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the "overlap map" $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth. (Since this map is automatically bijective and its inverse is a map of the same form with α and β interchanged, any such overlap map is a diffeomorphism.)

   Call a mwb atlas $\mathcal{A}$ on a topological space M *maximal* if $\mathcal{A}$ is not properly contained in another mwb atlas for $\mathcal{A}$. (Equivalently: call two mwb atlases on M *compatible* if their union is a mwb atlas; note that a necessary condition for compatibility is that the two given atlases have the same dimension. Compatibility is an equivalence relation on the set of mwb atlases of M. A maximal mwb atlas is the union of all mwb atlases compatible with some given mwb atlas.)

4. An n-dimensional *manifold with boundary* is a topological space M together with a maximal mwb atlas on M. Generally we simply call M a manifold-with-boundary, with the understanding that there is a maximal mwb atlas in the background that we don't wish to incorporate into the notation.

Henceforth, for a given manifold-with-boundary, we will simply call a mwb chart a *chart*.

We define "smooth map" from a manifold-with-boundary M to a manifold (or manifold-with-boundary) N just as we did for maps from a *manifold* M to a manifold N: we require the chart-representatives to be smooth maps from open subsets of $\mathbf{R}_+^{\dim(M)}$ to $\mathbf{R}^{\dim(N)}$ (or $\mathbf{R}_+^{\dim(N)}$).

Definition 1.2 Let M be an n-dimensional manifold-with-boundary. We call $p \in M$ a *boundary point* of M if, within the given maximal mwb atlas, there is a chart (U, ϕ) for which $\phi(p) \in \partial \mathbf{R}^n_+ = \mathbf{R}^{n-1} \times \{0\}$. The *boundary* of M (in the sense of "manifolds with boundary") is the set of boundary points of M, and is denoted ∂M. The *interior* of M is $M \setminus \partial M$.

Note that a manifold-with-boundary M is a (true) manifold iff $\partial M = \emptyset$.

Remark 1.3 In the *topological* sense of boundary (a notion defined for subsets of a given topological space), the boundary of every topological space is *empty*. The notation "∂M" for manifolds-with-boundary does *not* represent the topological boundary of M, unless (perhaps) M comes to us as a subset of some larger topological space, e.g. if M is a closed domain with regular boundary in a larger manifold (see part (c) below).

(a) Suppose $p \in \partial M$, and let (U, ϕ) be a chart of M for which $p \in U$ and $\phi(p) \in \partial \mathbf{R}^n_+$. Let (V, ψ) be another chart of M for which $p \in V$. Show that $\psi(p) \in \partial \mathbf{R}^n_+$ as well.

Thus, for any point p in M, either $\phi(p) \in \partial \mathbf{R}^n_+$ for *every* chart (U, ϕ) with $p \in U$, or for *no* chart (U, ϕ) with $p \in U$.

(There are essentially two ways to do this problem-part. One does not involve differentiability of overlap-maps at all; it simply uses the fact that they are homeomorphisms and applies a fact called *invariance of domain*, a nontrivial result you may have learned in a topology class. The other approach relies on the fact that our overlap-maps are assumed smooth (C^1 would be enough) and applies the Inverse Function Theorem.)

(b) Show that, if $\partial M \neq \emptyset$, then an n-dimensional mwb atlas on M determines, canonically, an $(n-1)$-dimensional *manifold* atlas on ∂M.

Set-up for remainder of problem. Just as for a manifold, an n-dimensional mwb has a *tangent space* at every point, an entire n-dimensional vector space even at a boundary point. I didn't define tangent spaces for a mwb in the initial problem-setup since the notion was not needed for the problem-parts above. But to generalize Stokes's Theorem from *domains with regular boundary* to *manifolds with boundary*, a definition of tangent space is needed for manifolds with boundary. Once tangent spaces of a mwb M are defined, the definitions of *vector field*, *cotangent space*, *differential form*, *orientation*, *orientability*, *Riemannian metric*, and the operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ are then the same as for a manifold.

At a boundary point p of a mwb M, to get the full tangent space T_pM we can't just use velocity vectors $\gamma'(0)$ of curves with $\gamma(0) = p$, since these vectors don't have additive inverses; intuitively, they form only a half-space in the yet-to-be-defined tangent space. One definition of *tangent bundle* and *tangent space* that works equally well for manifolds and mwb's, and generalizes usefully to other settings, is the following.

Definition 1.4 Let M be an n-dimensional mwb and let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be an atlas for M. Let $B = \{(p,\alpha) \in M \times A : p \in U_\alpha\}$. For each $(p,\alpha) \in B$ let

$$V_{p,\alpha} = \{(p,\alpha,v) : v \in \mathbf{R}^n\} = \{(p,\alpha)\} \times \mathbf{R}^n \quad \subset \; B \times \mathbf{R}^n.$$

Then let

$$\widetilde{TM} = \coprod\nolimits_{\{(p,\alpha)\in M\times A \,|\, p\in U_\alpha\}} V_{p,\alpha} \quad \subset \; M \times A \times \mathbf{R}^n$$

and define a relation $\sim$ on $\widetilde{TM}$ by

$$(p,\alpha,v) \sim (q,\beta,w) \iff p = q \in U_\alpha \cap U_\beta \text{ and } w = J_{\phi_\beta\circ\phi_\alpha^{-1}}(\phi_\alpha(p))v. \tag{1.7}$$

(As usual, "$\phi_\beta \circ \phi_\alpha^{-1}$" is short-hand for "$\phi_\beta \circ (\phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha\cap U_\beta)})$".) As a set, the *tangent bundle* TM is then defined to be $\widetilde{TM}/\sim$. The projections $V_{p,\alpha} \to U_\alpha$ given by $(p,\alpha,v) \mapsto p$ determine a well-defined surjective map $\pi : TM \to M$, and for each $p \in M$ we define the set T_pM to be $\pi^{-1}(\{p\})$.

For each $(p,\alpha) \in B$, the vector-space structure on $\mathbf{R}^n$ defines a vector-space structure on $\{(p,\alpha)\} \times \mathbf{R}^n$; the operations and the zero-element just ignore the "(p,α)". Observe that

$$(p,\alpha,0_{\mathbf{R}^n}) \sim (p,\beta,0_{\mathbf{R}_n}) \tag{1.8}$$

for $p \in U_\alpha \cap U_\beta$. Given such p,α,β and $v_1,v_2,w_1,w_2 \in \mathbf{R}^n$ such that $(p,\alpha,v_i) \sim (p,\beta,w_i)$ for $i \in \{1,2\}$, and given any $c_1,c_2 \in \mathbf{R}$, we have

$$\begin{aligned} c_1(p,\beta,w_1) + c_2(p,\beta,w_2) &:= (p,\beta,c_1w_1 + c_2w_2) && (1.9)\\ &\sim (p,\alpha,c_1v_1 + v_2v_2) \qquad \text{(by (1.7))} \\ &=: c_1(p,\alpha,v_1) + c_2(p,\alpha,v_2). && (1.10) \end{aligned}$$

(where the addition and scalar-multiplication operations on the LHS of (1.9) implicitly carry a subscript "(p,β)", and those on (1.10) implicitly carry a subscript "(p,α)").

Hence, for $p \in M$, we can define a vector-space structure on T_pM by selecting any α for which $p \in U_\alpha$, and using the vector-space structure on $\{(p,\alpha)\} \times \mathbf{R}^n$ defined above; equations (1.8) and (1.9)–(1.10) show that this algebraic structure on T_pM is independent of the choice of α.

Furthermore, the definition of *chart* for a mwb ensures that if $p \in \partial M$ and $(p,\alpha,v) \sim (p,\beta,w)$, then $v \in \mathbf{R}^n_+ \iff w \in \mathbf{R}^n_+$, and $v \in \text{interior}(\mathbf{R}^n_+) \iff w \in \text{interior}(\mathbf{R}^n_+)$. If $v \in \text{interior}(\mathbf{R}^n_+)$, we call the corresponding element of T_pM an *inward-pointing* tangent vector; we call $w \in T_pM$ an *outward-pointing* vector if $-w$ is inward-pointing.

(c) Let M be an n-dimensional mwb. For $p \in \partial M$ and $v \in T_pM$, check that v is inward-pointing iff $v = \gamma'(0)$ for some curve γ in M with $\gamma(0) = p$ and $\gamma(t) \in \text{interior}(M)$ for $t > 0$.

(d) Check that the definition we gave for $\int_D \omega$, where D is a domain with regular boundary in an oriented n-dimensional manifold M and $\omega \in \Omega_c^n(M)$, generalizes to a definition of $\int_M \omega$, where M is an oriented n-dimensional mwb and $\omega \in \Omega_c^n(M)$.

(e) Check that the definition we gave of *induced orientation on the boundary* of a domain with regular boundary in an oriented manifold M, generalizes to a definition of *induced orientation on* ∂M for an oriented mwb M.

(f) Check that (the proof we gave of) Stokes's Theorem generalizes to (a proof of) the analogous theorem for manifolds with boundary: if M^n is an oriented mwb, and ∂M is given the induced orientation, and $\omega \in \Omega_c^{n-1}(M)$, then

$$\int_M d\omega = \int_{\partial M} \omega := \int_{\partial M} j^*\omega,$$

where $j : \partial M \to M$ is the inclusion map.