

**Differential Geometry—MTG 6257—Spring 2026**  
**Problem Set 3**  
**Due-date: Friday, Apr. 3**

**Required reading:** problems 1–5 (including any extra reading that a problem’s instructions say to do), plus the posted notes “ $\mathcal{F}$ -linearity, tensoriality, and related notions.” Some problems introduce terminology and ideas used or needed in later problems.

**Required problems (to be handed in): 1ab, 2b(i), 2c, 3bc, 4c, 5.** In doing any of these problems, you may assume the results of all earlier problems (optional or required). You may also use results from notes I’ve posted for this class (in particular, the  $\mathcal{F}$ -linearity notes and the notes on tensor products), unless what I’m asking you to prove is one of those results whose proof was left as an exercise.

**Optional problems:** All the ones that are not required.

1. **Induced connections on dual bundles.** Let  $\nabla$  be a connection on a vector bundle  $E$  over a manifold  $M$ .

(a) Show that there is a unique connection  $\nabla'$  on the dual bundle  $E^*$  such that

$$X(\langle \xi, s \rangle) = \langle \nabla'_X \xi, s \rangle + \langle \xi, \nabla_X s \rangle \quad (1)$$

for all vector fields  $X$  and all sections  $\xi, s$  of  $E^*, E$  respectively. (In (1), the dual-pairings are taken pointwise, of course.)

*Strategy:* Rewrite equation (1) as  $\langle \nabla'_X \xi, s \rangle = X(\langle \xi, s \rangle) - \langle \xi, \nabla_X s \rangle$ , and then use a strategy similar to the proof of existence and uniqueness of the Levi-Civita connection on a Riemannian manifold. (Show that (i) for all  $X$  and  $\xi$ , the RHS can be used to define a section “ $\nabla'_X \xi$ ” of  $E^*$ , and then show that (ii) the map  $(X, \xi) \mapsto \nabla'_X \xi$  satisfies the properties of a connection.)

(b) Let  $\{s_\alpha\}_{\alpha=1}^k$  be a local basis of sections of  $E$  and let  $\{\xi^\alpha\}$  be the local basis of sections of  $E^*$  dual to  $\{s_\alpha\}$ . Let  $\Theta, \Theta' \in \Omega^1(U; M_{k \times k}(\mathbf{R}))$  be the connection forms of  $\nabla, \nabla'$  with respect to these local bases. (Here and in later problems,  $M_{k \times k}(\mathbf{R})$  denotes the space of  $k \times k$  real matrices.) Since we are using upper indices for the basis sections of  $E^*$ , we write the first index of  $\Theta'$  downstairs and the second index upstairs<sup>1</sup>:

$$\nabla' \xi^\beta = \xi^\alpha \otimes (\Theta')_{\alpha}{}^{\beta}$$

Show that  $\Theta'$  is the negative transpose of  $\Theta$ , in the sense that  $(\Theta')_{\alpha}{}^{\beta} = -\Theta^{\beta}{}_{\alpha}$ .

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<sup>1</sup>By default, LaTeX stacks superscripts directly on top of subscripts, as in  $B_j^k$ , making it impossible to distinguish which is the first index and which is the second. One way to produce, say,  $B^i_j$  in the pdf output, is to type  $\$ \{ B^i \}_j \$$  in the tex file.

(c) Show that for all  $p \in M$  and  $X, Y \in T_p M$ , the endomorphism  $F^{\nabla'}(X, Y) : E_p^* \rightarrow E_p^*$  is the negative of the natural adjoint of  $F^{\nabla}(X, Y) : E_p \rightarrow E_p$ . (For the definition of “natural adjoint”, see the Appendix in the posted tensor-product notes.)

**2. Induced connections on direct sums, tensor products, and homomorphism bundles.** Let  $\nabla^{(1)}, \nabla^{(2)}$  be connections on vector bundles  $E_1, E_2$  over a manifold  $M$ . Let  $U \subset M$  be an open set over which both  $E_1$  and  $E_2$  are trivial, let  $\mathbf{s} := \{s_\alpha\}_{\alpha=1}^{k_1}, \mathbf{t} := \{t_\mu\}_{\mu=1}^{k_2}$  be bases of sections of  $E_1, E_2$  (respectively) over  $U$ , and let  $\Theta^{(1)} \in \Omega^1(U; M_{k_1 \times k_1}(\mathbf{R})), \Theta^{(2)} \in \Omega^1(U; M_{k_2 \times k_2}(\mathbf{R}))$  be the corresponding connection forms. (Your answer should be expressed in terms of  $\Theta^{(1)}$  and  $\Theta^{(2)}$ .)

(a) The *direct sum connection*  $\nabla$  on  $E_1 \oplus E_2$  is defined by  $\nabla \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \nabla^{(1)} s \\ \nabla^{(2)} t \end{pmatrix}$ , i.e.  $\nabla_X \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \nabla_X^{(1)} s \\ \nabla_X^{(2)} t \end{pmatrix}$ , where  $s \in \Gamma(E_1), t \in \Gamma(E_2)$ , and  $X \in \Gamma(TM)$ . (It is easy to check that this *does* define a connection.) Find the connection form of  $\nabla$  with respect to the basis  $\begin{pmatrix} s_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} s_{k_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ t_{k_2} \end{pmatrix}$ .

(b) (i) Show that there is a unique connection  $\nabla$  on  $E_1 \otimes E_2$  such that

$$\nabla_X(s \otimes t) = (\nabla_X^{(1)} s) \otimes t + s \otimes \nabla_X^{(2)} t \quad (2)$$

for all  $s \in \Gamma(E_1), t \in \Gamma(E_2)$ , and  $X \in \Gamma(TM)$ . Note that this cannot be deduced from applying the universal property of tensor products to  $\Gamma(E_1) \otimes \Gamma(E_2)$ , since “ $s \otimes t$ ” denotes the *pointwise* tensor product  $p \mapsto s(p) \otimes t(p)$ . If we attempt to take (2) as a *definition* of  $\nabla$ , it is not obvious without a little computation that  $\nabla_X(s \otimes t)$  is well-defined, not just because elements of the form  $s_p \otimes t_p$  don’t form a basis of the vector space  $E_{1,p} \otimes E_{2,p}$ , but because  $s \otimes t = fs \otimes (1/f)t$  for any nonvanishing  $f \in C^\infty(M)$ .

We call  $\nabla$  the *tensor product connection* determined by  $\nabla^{(1)}$  and  $\nabla^{(2)}$ .

(ii) Find the connection form of  $\nabla$  with respect to the local basis of sections  $\{s_\alpha \otimes t_\mu\}$  of  $E_1 \otimes E_2$ , expressed in terms of  $\Theta^{(1)}$  and  $\Theta^{(2)}$ . To write this down efficiently, it’s helpful to have an object you can call the “tensor product of two matrices”. This object is defined in the handout “Some notes on tensor products”; see the Remark in these notes entitled “‘tensor product’ of two matrices”. (As of this writing, this Remark resides in Section 2.5, “The finite-dimensional case”, as Remark 2.33. Numbering in these notes is subject to change, since the notes are still a work in progress.) The precise definition is really not important for this exercise; all that really matters is that there *is* a definition that’s consistent with standard conventions of linear algebra.

(iii) Show that the curvature  $F^\nabla$  satisfies

$$F^\nabla(X, Y)(s \otimes t) = (F^{\nabla^{(1)}}(X, Y)s) \otimes t + s \otimes (F^{\nabla^{(2)}}(X, Y)t) \quad (3)$$

for all  $s \in \Gamma(E_1), t \in \Gamma(E_2)$ , and  $X, Y \in \Gamma(TM)$ . We may write (3) symbolically as

$$F^\nabla = F^{\nabla^{(1)}} \otimes \text{id}_{E_2} + \text{id}_{E_1} \otimes F^{\nabla^{(2)}}.$$

(c) Combining problems 2b and 1, there is an induced connection  $\nabla$  on  $E_2 \otimes E_1^* \cong_{\text{canon.}} \text{Hom}(E_1, E_2)$ . Show that this connection satisfies

$$(\nabla_X A)(s) = \nabla_X^{(2)}(A(s)) - A(\nabla_X^{(1)}(s))$$

for all  $A \in \Gamma(\text{Hom}(E_1, E_2))$ ,  $s \in \Gamma(E_1)$ , and  $X \in \Gamma(TM)$ . Note that this can be written as the Leibnizian-looking formula

$$\nabla_X^{(2)}(A(s)) = (\nabla_X A)(s) + A(\nabla_X^{(1)}(s)).$$

(d) Show that the induced connection  $\nabla$  on  $\text{End}(E_1) \cong_{\text{can.}} E_1 \otimes E_1^*$  satisfies

$$F^\nabla(X, Y)A = [F^{\nabla^{(1)}}(X, Y), A] := F^{\nabla^{(1)}}(X, Y) \circ A - A \circ F^{\nabla^{(1)}}(X, Y)$$

for all  $A \in \Gamma(\text{End}(E_1))$  and  $X, Y \in \Gamma(TM)$ .

(e) Let  $\mathbf{R}_M$  denote the product bundle  $M \times \mathbf{R} \rightarrow M$ , and let  $\nabla^{(0)}$  denote the canonical connection on  $\mathbf{R}_M$ . (This is the connection I called the “canonical flat connection” in class. Recall from class that  $\nabla^{(0)}f = df$ .) The bundles  $\mathbf{R}_M \otimes E_1$  and  $E_1 \otimes \mathbf{R}_M$  are canonically isomorphic to  $E_1$ . Show that these canonical isomorphisms carry the tensor-product connections on  $\mathbf{R}_M \otimes E_1$  and  $E_1 \otimes \mathbf{R}_M$ , induced by  $\nabla^{(0)}$  and  $\nabla^{(1)}$ , to the connection  $\nabla^{(1)}$  on  $E_1$ .

**Remark 1 (A convention used below)** The construction of direct-sum connections and tensor-product connections extends in an obvious way to direct sums and tensor products of more than two vector bundles. In particular, a connection on a vector bundle  $E$  induces a connection on any bundle of the form  $E_1 \otimes \dots \otimes E_k$  ( $k \geq 1$ ) where for each  $i$ , the bundle  $E_i$  is either  $E$  or  $E^*$ . When considering the whole collection of such tensor-product bundles (“built” from a single vector bundle  $E$ ), or even just a few such bundles, it is usually too cumbersome to use distinct notation for each of these induced connections (as we did in problems 1 and 2). Hence, if  $\nabla$  is a connection on  $E$ , we generally use the same notation  $\nabla$  for the induced connection on any of these bundles. In any term in a formula or equation, context—the type of section being differentiated—makes clear which connection is being used.  $\blacktriangle$

3. Let  $\nabla$  be a connection on a vector bundle  $E$  over a manifold  $M$ . By problems 1 and 2,  $\nabla$  induces a connection on  $E^* \otimes E^*$ .

(a) Show that the induced connection  $\nabla$  on  $E^* \otimes E^*$  preserves the sub-bundles  $\text{Sym}^2(E^*)$  and  $\wedge^2(E^*)$  in the following sense: if  $s$  is a section of either of these sub-bundles, and  $X$  is a vector field on  $M$ , then  $\nabla_X s$  is a section of the same sub-bundle. Thus, the restriction of  $\nabla$  to sections of either of these sub-bundles is a connection on that sub-bundle.

(b) Let  $g$  be a Riemannian metric (in the vector-bundle sense) on  $E$ . Then  $g$  is an element of  $\Gamma(E^* \otimes E^*)$  (in fact,  $g$  is an element of  $\Gamma(\text{Sym}^2(E^*))$ , but this will not be needed below), so  $\nabla g$  is defined. Show that the connection  $\nabla$  on  $E$  respects  $g$  (as defined in class) if and only if  $\nabla g = 0$  (whence the terminology “ $\nabla$  preserves  $g$ ”).

In particular: on any Riemannian manifold, the Riemannian metric is covariantly constant with respect to the Levi-Civita connection. (The zero-torsion criterion is not needed for this, except to define what the Levi-Civita connection is!)

(c) In the setting of part (b), let  $\mathbf{g} \in \Gamma(\text{Hom}(E, E^*))$  be the section whose value at  $p \in M$  is the isomorphism  $\mathbf{g}_p : E_p \rightarrow E_p^*$  determined by the metric  $g$ . Show that if  $g$  is covariantly constant (i.e.  $\nabla g = 0$ ), then so is  $\mathbf{g}$ :

$$\nabla_X(\mathbf{g}(s)) = \mathbf{g}(\nabla_X s) \quad \text{for all } s \in \Gamma(E), X \in \Gamma(TM). \quad (4)$$

(Problem 2c shows why statement (4) is equivalent to “ $\nabla_X \mathbf{g} = 0$  for all  $X$ ,” hence to  $\nabla \mathbf{g} = 0$ .)

**4. The covariant Hessian.** Let  $\nabla^E$  be a connection on a vector bundle  $E$  over  $M$ , and let  $\nabla^M$  be a connection on  $TM$ . For all vector fields  $X, Y$  and all  $s \in \Gamma(E)$ , let

$$(\tilde{H}s)(X, Y) = \nabla_X^E \nabla_Y^E s - \nabla_{\nabla_X^M Y}^E s.$$

(In case the last term of the formula is hard to read: in that term, “ $\nabla_X^M Y$ ” is a subscript to  $\nabla^E$ ; at each  $p \in M$  we are differentiating  $s$  in the direction  $\nabla_X^M Y|_p$ , using the connection  $\nabla^E$ .)

(a) Show that  $(\tilde{H}s)(X, Y)$  is  $\mathcal{F}$ -bilinear in  $(X, Y)$ . Hence, for each  $s$ , the map  $(X, Y) \mapsto (\tilde{H}s)(X, Y)$  is tensorial, and therefore defines a section  $Hs$  of  $E \otimes T^*M \otimes T^*M$ .

The section  $Hs$  is called the *covariant Hessian of  $s$*  with respect to the connections  $\nabla^E$  and  $\nabla^M$ . If  $(M, g)$  is Riemannian, and you see the term “covariant Hessian” used without the connections  $\nabla^E$  and  $\nabla^M$  having both been specified explicitly, the writer is probably using the following conventions:

- $\nabla^M$  is the Levi-Civita connection on  $(M, g)$ .
- If  $E$  is a tensor bundle over  $M$  (see Remark 2 below), then  $\nabla^E$  is the one induced by the Levi-Civita connection. Note that for the product bundle  $\mathbf{R}_M$ ,

this means that the canonical connection ( $\nabla f = df$ ) is used, so the covariant Hessian of  $f \in C^\infty(M)$  is given by  $Hf(X, Y) = X(Y(f)) - (\nabla_X Y)(f)$ , where  $\nabla$  is the Levi-Civita connection.

(b) Show that if  $\nabla^M$  is torsion-free, then for all sections  $s$  and vector fields  $X, Y$ ,

$$Hs(X, Y) - Hs(Y, X) = F^{\nabla^E}(X, Y)s.$$

Thus, the left-hand side is tensorial in  $s$ , even though neither term individually is tensorial in  $s$ .

(c) Following the conventions mentioned above, show that on a Riemannian manifold, the covariant Hessian of any function  $f \in C^\infty(M)$  is a *symmetric* tensor field. (All that is needed for this symmetry is that we use a torsion-free connection on  $TM$ ; the metric does not enter the argument.)

**Remark 2 (A further convention)** Given a manifold  $M$ , let us use the term *tensor bundle* (over  $M$ ) for any bundle of the form  $E_1 \otimes \dots \otimes E_k$  ( $k \geq 1$ ), where for each  $i$ , the bundle  $E_i$  is either  $TM$  or  $T^*M$ ; if  $k = 1$  we also allow the trivial product bundle  $\mathbf{R}_M = M \times \mathbf{R} \rightarrow M$ .<sup>2</sup> (Because of the canonical isomorphisms mentioned in problem 2(e), we gain no new bundles by allowing  $E_i = \mathbf{R}_M$  if  $k > 1$ , but there is no harm in allowing it.) A connection  $\nabla$  on  $TM$  then induces a connection (also denoted  $\nabla$ ) on every tensor bundle over  $M$ , provided we define which connection to use on the trivial bundle  $\mathbf{R}_M$ . In view of problem 2(e), in the context of induced connections on tensor bundles, we define the “induced” connection  $\nabla$  on  $\mathbf{R}_M$  to be the canonical connection on this product bundle (no matter what connection is used on  $TM$ ).

With this convention, given a connection  $\nabla$  on  $TM$ , the collection of induced connections on tensor bundles is “Leibnizian with respect to contractions” in the sense that equation (1) holds with a single symbol “ $\nabla$ ”, and “Leibnizian with respect to tensor products” in the sense that equation (2) holds with a single symbol “ $\nabla$ ”.  $\blacktriangle$

**5. Higher-order covariant derivatives.** Let  $\nabla$  be a connection on  $TM$ . Using the convention in Remark 2, we have an induced connection denoted  $\nabla$  on every tensor bundle over  $M$ . Since a connection on any vector bundle  $E$  maps  $\Gamma(E)$  to  $\Gamma(E \otimes T^*M)$ , we therefore have an infinite sequence of maps

$$C^\infty(M) = \Gamma(\mathbf{R}_M) \xrightarrow{\nabla=d} \Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes T^*M) \xrightarrow{\nabla} \dots \quad (5)$$

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<sup>2</sup>More generally, the term “tensor bundle” may be used for sub-bundles and direct sums of what we called tensor bundles above. However, for simplicity’s sake, in this assignment we will stick to the narrower convention in which “tensor bundle” means *only* a bundle of the type in Remark 2.

(For any tensor bundle  $E$  we have a similar sequence, with  $\Gamma(\mathbf{R}_M)$  replaced by  $\Gamma(E)$ , and with  $(T^*M)^{\otimes k}$  replaced by  $E \otimes (T^*M)^{\otimes k}$ .) In particular, for  $f \in C^\infty(M)$ , a connection on  $TM$  allows us to define  $\nabla\nabla f, \nabla\nabla\nabla f$ , etc.

(a) Show that if the connection  $\nabla$  is torsion-free, then  $\nabla\nabla f$  is the covariant Hessian defined in problem 4.

If you do part (a) carefully, you will likely find that you are actually doing part (b), and then deducing part (a) using problem part (b). I've stated part (a) separately since it's more forgiving of an easily-made mistake.

(b) More generally, show that if the connection  $\nabla$  is arbitrary, then  $\nabla\nabla f$  is still the covariant Hessian up to a "transpose":

$$(\nabla\nabla f)(X, Y) = Hf(Y, X). \quad (6)$$

(This is true with  $f$  replaced by a section of any tensor bundle; I'm just giving you the simplest case in this assignment. An analog is also true for sections of an *arbitrary* vector bundle  $E$ , except that we need to specify two initial connections,  $\nabla^E$  and  $\nabla^M$ , to define what " $\nabla$ " is going to mean beyond the first map in the sequence analogous to (5).)

**6. Covariant exterior derivative.** Let  $E$  be a vector bundle over a manifold  $M$ . We will use the abbreviated notation " $\Omega^j(E)$ " for  $\Omega^j(M; E) = \Gamma(E \otimes \wedge^j T^*M)$ .

(a) Let  $j, l \geq 0$ .

(i) Show that there is a unique bilinear map  $\wedge : \Omega^j(E) \times \Omega^l(M) \rightarrow \Omega^{j+l}(E)$ ,  $(\alpha, \omega) \mapsto \alpha \wedge \omega$ , satisfying

$$(s \otimes \eta)_p \wedge \omega_p = s_p \otimes (\eta \wedge \omega)_p \quad \text{for all } p \in M. \quad (7)$$

(I only *asserted* this in class. As with most tensor-product-related definitions and existence/uniqueness statements—including all such items in this assignment—there are details that need to be checked, of types I've mentioned several times in class.)

(ii) Show that there is a unique bilinear map  $\Omega^j(\text{End}(E)) \times \Omega^l(E) \rightarrow \Omega^{j+l}(E)$  satisfying

$$((A \otimes \eta)_p, (s \otimes \omega)_p) \mapsto A_p(s_p) \otimes (\eta_p \wedge \omega_p) \quad \text{for all } p \in M. \quad (8)$$

(In this equation, the endomorphism  $A_p$  is applied to the vector  $s_p \in E_p$ , while the  $\wedge^* T_p^* M$ -factors are wedged together.) Henceforth we omit the subscript  $p$  equations like (7) and (8), understanding that an equation like " $(s \otimes \eta) \wedge \omega = s \otimes (\eta \wedge \omega)$ " is to be interpreted as a pointwise statement.

For  $F \in \Omega^j(\text{End}(E))$  and  $\xi \in \Omega^l(E)$ , we will write  $F(\xi)$  for the image of  $(F, \xi)$  under the map defined pointwise by (8). Regrettably, the notation is not self-explanatory, but I know of no wonderful notation for this combined endomorphism-evaluation/wedge-product operation.

(b) Let  $\nabla$  be a connection on  $E$ . Show that, for all  $j \geq 0$ , there is a unique linear map  $d_\nabla : \Omega^j(E) \rightarrow \Omega^{j+1}(E)$  that satisfies

$$d_\nabla(s \otimes \omega) = (\nabla s) \wedge \omega + s \otimes d\omega \quad (9)$$

for all  $s \in \Gamma(E)$ ,  $\omega \in \Omega^j(M)$ , and  $j \geq 0$ . We call the collection of operators  $d_\nabla$  the *covariant exterior derivative* operator determined by  $\nabla$ . (Just as for the ordinary exterior derivative, there is actually one operator for each  $j$ , but we use the same name and notation for all of them.)

(c) Notation as in (b). Show that, for  $j \geq 0$ , the operator  $d_\nabla : \Omega^j(E) \rightarrow \Omega^{j+1}(E)$  is not  $\mathcal{F}$ -linear, but that  $d_\nabla \circ d_\nabla : \Omega^j(E) \rightarrow \Omega^{j+2}(E)$  is  $\mathcal{F}$ -linear.

(d) Let  $F^\nabla \in \Omega^2(\text{End}(E))$  be the curvature 2-form of  $\nabla$ .

(i) Show that for every  $s \in \Gamma(E)$ ,  $d_\nabla d_\nabla s = F^\nabla(s)$ , where the notation is as in (a)(ii) above (with  $j = 0$ ).

(I sketched this in class on Wednesday 3/25/26, but most students were absent.)

(ii) Show, more generally, that for any  $j \geq 0$  and  $\xi \in \Omega^j(E)$ ,  $d_\nabla d_\nabla \xi = F^\nabla(\xi)$ .

**Remark 3** Hence for a *flat* connection (a connection whose curvature is zero), the pair  $(\Omega^*(E), d_\nabla)$  is a cochain complex, and cohomology is defined. However most vector bundles do not admit a flat connection; there are topological obstructions. When flat connections do exist, they are rarely unique, and the cohomology groups (in a given degree) defined by different flat connections need not be isomorphic.  $\blacktriangle$

**7. Torsion and the covariant exterior derivative.** Let  $M$  be a manifold. The identity map  $I : TM \rightarrow TM$  may be viewed as a  $TM$ -valued 1-form on  $M$ . (Note that for a general vector bundle, there is no analog of this special 1-form.)

Let  $\nabla$  be a connection on  $TM$ .

(a) Show that

$$d_\nabla I = \tau^\nabla, \quad (10)$$

where the torsion tensor-field  $\tau^\nabla$  is viewed as a  $TM$ -valued 2-form (just as is  $d_\nabla I$ ). I.e. *the torsion of a connection on  $TM$  is the covariant exterior derivative of the "identity 1-form"  $I \in \Omega^1(M; TM)$ .*

**Remark 4** Above, we treated  $I$  as an element of  $\Omega^1(M; TM)$ ; the object  $d_{\nabla}I$  was then an element of  $\Omega^2(M; TM)$ . But we may also view  $I$  as tensor field on  $M$ , a section of the bundle  $\text{End}(TM) = \text{End}(TM \otimes T^*M)$ . (In terms of bundle-valued differential forms,  $I$  is then a element of  $\Omega^0(M; \text{End}(TM))$  rather than  $\Omega^1(M; TM)$ .) From an earlier problem on this assignment, the connection  $\nabla$  on  $TM$  induces a connection on  $\text{End}(TM)$ . With this induced connection, treating  $I$  as a section of  $\text{End}(TM)$ , we have  $\nabla I = 0 \in \Gamma(\text{End}(TM) \otimes T^*M) = \Omega^1(M; \text{End}(TM))$ .  $\blacktriangle$

(b) Suppose  $\tau^{\nabla} = 0$ . Then, viewing  $I$  as a  $TM$ -valued 1-form, by part (a) we have  $d_{\nabla}d_{\nabla}I = d_{\nabla}(0) = 0 \in \Omega^3(M; TM)$ . But by problem 6d(ii), we also have  $d_{\nabla}d_{\nabla}I = F^{\nabla}(I) \in \Omega^3(M; TM)$ . Hence  $F^{\nabla}(I) = 0$ .

In particular “ $F^{\nabla}(I) = 0$ ” holds if  $\nabla$  is the Levi-Civita connection for a Riemannian metric  $g$ . Use the definition of  $F^{\nabla}(I)$  (plus the equation  $F^{\nabla}(I) = 0$ ) to derive a symmetry of the Riemann tensor that we have derived (or will derive soon) by other means.