# Differential Geometry—MTG 6257—Spring 2026
# Problem Set 4
## Due-date: Wednesday, 4/22/26

**Required reading**: All the problems, and the handout "Pullbacks of vector bundles and connections".

**Required problems (to be handed in)**: 1, 2, 3bc, 4, 5ac.

In doing any of these problem parts, you may assume the results of all earlier problems and problem-parts (optional or required).

**Optional problems**: All the ones that are not required.

Notation for all the problems: unless otherwise specified, $M$ is an $n$-dimensional manifold. (Some problems may still state this explicitly.)

**1.** Let $(M, g) = (\mathbf{R}^3, g_{\rm std})$ and let $(x, y, z)$ denote the standard coordinates. Let $U \subset \mathbf{R}^2$ be an open neighborhood of $(0, 0)$, let $f : U \to \mathbf{R}$, and let $Q \subset \mathbf{R}^3$ be the graph of $f$. Assume that $f(0, 0) = 0$ and that $df|_{(0,0)} = 0$, so that the plane in $\mathbf{R}^3$ tangent to $Q$ at $(0, 0, 0)$ (the "embedded tangent space" at the origin) is the $xy$ plane. Let $N$ be the upward-pointing normal vector field (i.e. the one whose $z$-component is positive). Let $\bar{h}$ be the scalar second fundamental form of $Q$ determined by $N$.

The Taylor expansion of $f$ near $(0, 0)$ is of the form $f(x, y) = \frac{1}{2}(ax^2 + 2bxy + cy^2) +$ (higher-order remainder). Express $\bar{h}$ at the origin in terms of $a, b$, and $c$.

Figure out how this generalizes to graphs of functions $f : (U \subset \mathbf{R}^n) \to \mathbf{R}$.

**Remark**. For any hypersurface (:= codimension-one submanifold) $Q \subset \mathbf{R}^n$, and any $p \in Q$, we can rotate and translate the coordinate axes in $\mathbf{R}^n$ to make $p$ the origin and $T_pQ$ the hyperplane $H = \{(x^1, \ldots, x^n) \in \mathbf{R}^n \mid x^n = 0\}$. The Implicit Function Theorem then implies that, near the origin, $Q$ is the graph of a function $f : (U \subset H) \to \mathbf{R}$. So, your work above provides a general interpretation of the second fundamental form of a hypersurface in $\mathbf{R}^n$: it describes the second-order deviation of the surface from its tangent plane at any point. (There *is* no first-order deviation; that's what "tangent plane" means.) This is one reason the second fundamental form of a submanifold of $\mathbf{R}^n$ is often called the "extrinsic curvature" of the submanifold; it's something that an observer in $\mathbf{R}^n$, *external* to the submanifold, might describe as "curvature." The Riemann tensor of a submanifold of $\mathbf{R}^n$ is thought of as "intrinsic curvature": once one has the metric on $Q$, nothing involving the ambient manifold is needed to define or compute the Riemann tensor.

**2. Lemma for use in next problem.** Let $\{y^i\}$ be standard coordinates on $\mathbf{R}^n$, let $\omega \in \Omega^{n-1}(S^{n-1})$ be the standard volume form and let $\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} \omega$ (the volume of the standard, Euclidean, unit sphere).[1] Show that for all $i, j \in \{1, \dots, n\}$,

$$\int_{S^{n-1}} y^i y^j \ \omega = \frac{1}{n}\, \delta_{ij}\, \mathrm{Vol}(S^{n-1}).$$

(This can be done without any trigonometric integrals.)

*Note:* $\mathrm{Vol}(S^{n-1})$ can be computed explicitly. I simply am not asking you to do the computation. (However, it can be reduced to a trigonometric integral of the type we teach "reduction formulas" for in Calc 1.)

**3. Ricci tensor and scalar curvature.** Let $(M, g)$ be a Riemannian manifold. For each $p \in M$ and $X, Y \in T_pM$, the Riemann tensor determines a linear map $T_pM \to T_pM$ defined by $Z \mapsto R(X, Z)Y$. Define

$$\mathsf{Ric}(X, Y) = \mathsf{Ric}|_p(X, Y) = \mathrm{tr}\big(Z \mapsto R(X, Z)Y\big),$$

where "tr" denotes the trace. Thus, if $\{e_i\}$ is an arbitrary basis of $T_pM$ and $\{\theta^i\}$ is the dual basis of $T_p^*M$,

$$\mathsf{Ric}(X, Y) = \langle \theta^i, R(X, e_i)Y \rangle.$$

Clearly the map $(X, Y) \mapsto \mathsf{Ric}|_p(X, Y)$ is bilinear, so $\mathsf{Ric}|_p$ is an element of $T_p^*M \otimes T_p^*M$. This bilinear form is called the *Ricci tensor* at $p$. Letting $p$ vary, it is easily seen that $\mathsf{Ric}|_p$ depends smoothly on $p$, so $\mathsf{Ric}$ becomes a tensor field on $M$, called the *Ricci tensor* (field) or the *Ricci curvature.*

(a) Show that with $p, \{e_i\}, \{\theta^i\}$ as above, the Ricci tensor at $p$ is given by

$$\begin{aligned} \mathsf{Ric} \ &= \ R_{jl}\, \theta^j \otimes \theta^l, \\ &\quad\ \text{where } R_{jl} = R^i{}_{jil} \end{aligned}$$

and where $\{R^i{}_{jkl}\}$ are the components of the Riemann tensor at $p$ with respect to the given bases.

(b) Show that the Ricci tensor is a *symmetric* tensor field: for all $p \in M$ and all $X, Y \in T_pM$, we have $\mathsf{Ric}(X, Y) = \mathsf{Ric}(Y, X)$.

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[1] You will not need a formula for $\omega$ to do this problem, but as a reminder, one definition is $\omega = j^*\big(\iota_N(dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n)\big)$, where $j : S^{n-1} \hookrightarrow \mathbf{R}^n$ is the inclusion map and $N = \sum_i y^i \frac{\partial}{\partial y^i}$, a vector field on $\mathbf{R}^n$ that restricts to the outward-pointing unit normal vector field along $S^{n-1}$.

Suggestion: Compute the trace defining $\mathsf{Ric}(X, Y)$ using an orthonormal basis of $T_pM$. The dual pairing with $\theta^i$ then becomes inner product with $e_i$.

(c) Below, for any normed vector space $V$, we write $S(V)$ for the unit sphere centered at the origin.

Assume that $n = \dim(M) \geq 2$. Recall that, at each $p$, the sectional curvature of $M$ at $p$ is a map $G_2(T_pM) \to \mathbf{R}$, $\mathcal{P} \mapsto \sigma(\mathcal{P})$. For $X \in S(T_pM)$ let $X^\perp = \{Y \in T_pM : Y \perp X\}$. Let $G_2^X(T_pM) \subset G_2(T_pM)$ denote the set of all 2-planes in $T_pM$ that contain $X$. There is a two-to-one map

$$\begin{aligned} \pi_X : S(X^\perp) &\to G_2^X(T_pM), \\ \pi_X(Y) &= \mathcal{P}(X, Y) := \operatorname{span}\{X, Y\}. \end{aligned}$$

(The "two-to-one" comes from the fact that $\pi_X(-Y) = \pi_X(Y)$.) The vector space $X^\perp$ is a Riemannian manifold with the standard Riemannian metric determined by $g_p|_{X^\perp}$; thus $S(X^\perp)$ inherits a Riemannian metric. Orienting $X^\perp$ arbitrarily, and giving $S^{n-1}$ the induced orientation, we then obtain a volume form form $\omega_{n-2}$ on $S(X^\perp)$. (The subscript here is just a reminder of the dimension of $S(X^\perp)$.) Show that for $X \in S(T_pM)$,

$$\int_{S(X^\perp)} (\sigma \circ \pi_X)\, \omega_{n-2} = \int_{S(X^\perp)} \sigma(\mathcal{P}(X, \cdot))\, \omega_{n-2} = \frac{\mathrm{Vol}(S^{n-2})}{n-1} \mathsf{Ric}(X, X), \tag{1}$$

and hence

$$\frac{1}{n-1} \mathsf{Ric}(X, X) = \frac{1}{\mathrm{Vol}(S(X^\perp))} \int_{S(X^\perp)} (\sigma \circ \pi_X)\, \omega_{n-2}\ . \tag{2}$$

(Thus, up to the normalization constant $\frac{1}{n-1}$, for each unit vector $X \in T_pM$ the quantity $\mathsf{Ric}(X, X)$ represents the *average sectional curvature among all two-planes in $T_pM$ that contain $X$*.)[2]

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[2] The reason we integrated over $S(X^\perp)$ in (1) and (2), rather than over $G_2^X(T_pM)$, is that $G_2^X(T_pM)$ is diffeomorphic to the projective space $\mathbf{RP}^{n-2}$, which is not orientable when $n$ is even. However, whether or not a Riemannian manifold $(N, g_N)$ is orientable, the metric $g_N$ induces a well-defined measure "$d\mu_N$" on $N$; it's simply something that we did not discuss in the non-orientable case (it's not a differential form in that case). Therefore for any finite-dimensional inner-product space $W$, the projectization $\mathbf{P}(W)$ has a Riemannian metric, hence Riemannian measure $d\mu$, induced the by the natural two-to-one covering map $\pi' : S(W) \to \mathbf{P}(W)$ and the standard Riemannian metric on $S(W)$. (Here we regard $W$ as a Riemannian manifold with the standard Riemannian metric determined by the given inner product on $W$.) Using these facts it can be shown $\mathrm{Vol}(S(X^\perp)) = 2\mathrm{Vol}(G_2^X(T_pM))$ and that

[FOOTNOTE CONTINUES ON NEXT PAGE]

**Remark 1 (covered in class)** Recall that for any finite-dimensional vector space $V$, any symmetric bilinear form $h : V \times V \to \mathbf{R}$ is determined by its restriction to the diagonal: if we know $h(X, X)$ for *all* $X \in V$, then we know $h(X, Y)$ for all $X, Y \in V$. This follows from the *polarization identity*

$$h(X,Y) = \frac{h(X+Y, X+Y) - h(X-Y, X-Y)}{4} \ .$$

Furthermore, if $V$ is equipped with a norm $\| \ \|$, then for all nonzero $X \in V$ we have $h(X, X) = \|X\|^2 h(\hat{X}, \hat{X})$, where $\hat{X} = X/\|X\|$. Thus, in the presence of a norm, a symmetric bilinear form $h$ can be completely recovered from the function $f_h$ (notation just for this problem) that $h$ determines on the unit sphere:

$$\begin{array}{rcl} f_h : S(V) := \{X \in V : \|X\| = 1\} & \to & \mathbf{R}, \\ X & \mapsto & f_h(X) := h(X, X). \end{array}$$

In particular, for each $p \in M$, the function $f_{\mathsf{Ric}} : S(T_pM) \subset T_pM$ carries all the information of the Ricci tensor at $p$.

(d) Let $\mathsf{g}_p : T_pM \to T_p^*M$ be the isomorphism induced by the inner product $g_p$. For any tensor $h_p \in T_p^*M \otimes T_p^*M$, we define the *trace of $h_p$ with respect to $g_p$*, denoted $\mathrm{tr}_{g_p}(h_p)$, to be the image of $h_p$ under the following composition

$$T_p^*M \otimes T_p^*M \overset{\mathsf{g_p}^{-1} \otimes \mathrm{id}}{\longrightarrow} T_pM \otimes T_p^*M \overset{\text{canon. iso.}}{\longrightarrow} \mathrm{Hom}(T_pM, T_pM) \overset{\text{trace}}{\longrightarrow} \mathbf{R}.$$

Applying this pointwise to any $h \in \Gamma(\mathrm{Sym}^2(T^*M))$ gives a real-valued function $\mathrm{tr}_g(h) : M \to \mathbf{R}$.

Show that for $h$ as above, $p \in M$, $\{e_i\}$ any basis of $T_pM$,

$$\mathrm{tr}_g(h)\big|_p = g^{ij} h_{ij} = h^i{}_i = h_i{}^i \ ,$$

where $h_{ij} = h(e_i, e_j)$, $g_{\bullet\bullet}$ is the matrix of $g_p$ with respect to the basis $\{e_i\}$ (i.e. $g_{ij} = g(e_i, e_j)$), and $g^{\bullet\bullet} = (g_{\bullet\bullet})^{-1}$.

(e) The *scalar curvature* or *Ricci scalar* is the real-valued function $\mathsf{R} = \mathrm{tr}_g(\mathsf{Ric})$ on $M$. Show that at each $p \in M$,

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$$\int_{S(X^\perp)} (\sigma \circ \pi_X) \ \omega = 2 \int_{G_2^X(T_pM)} \sigma \ d\mu.$$

Thus (2) indeed represents the average value of the function $\sigma|_{G_2^X(T_pM)}$ with respect to the induced measure on $G_2^X(T_pM)$.

$$\frac{1}{n}\mathsf{R}(p) = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{S(T_pM)} f_{\mathsf{Ric}}\ \omega_{n-1}\ ,$$

where $f_{\mathsf{Ric}}$ is as in Remark 1 and $\omega_{n-1}$ is the volume form on the sphere $S(T_pM)$ induced by the metric $g_p$ and an arbitrary choice of orientation of $T_pM$.

Thus, up to the normalization constant $\frac{1}{n}$, the scalar curvature at $p$ is the average value of the function $S(T_pM) \to \mathbf{R}, X \mapsto \mathsf{Ric}(X,X)$. But for each $X \in S(T_pM)$, the quantity $f_{\mathsf{Ric}}(X)$ is itself an average of sectional curvatures (up to a factor of $\frac{1}{n-1}$), so scalar curvature is sometimes thought of as a "double average" of sectional curvatures. However, the word "double" can be eliminated: it can be shown that, up to a dimensional constant, $\mathsf{R}(p)$ is simply the average value of the sectional-curvature function $\sigma_p : G_2(T_pM) \to \mathbf{R}$.

**4.** For $(M,g) = (S^n, g_{\mathrm{std}})$, compute the Ricci tensor and scalar curvature. (You should find that the Ricci tensor is a constant multiple of $g_{\mathrm{std}}$ and that the scalar curvature is constant.)

### 5. Connections on the pulled-back tangent bundle

*Before doing this problem you should read the notes "Pullbacks of Vector Bundles and Connections".* The notation "$F^\sharp$" (with "$F$" called "$f$") is defined in Section 2 of these notes.

Let $F : N \to M$ be a smooth map of manifolds. As discussed last semester, a vector field on $N$ does not, in general, push forward to a vector field on $M$. However, it *does* push forward to a section of the pulled-back tangent bundle: Given $X \in \Gamma(TN)$, we can define a section $\hat{X} \in \Gamma(F^*TM)$ by

$$\hat{X}_p := F_p^\sharp(F_{*p}X_p). \tag{3}$$

(a) Let $\nabla'$ be an arbitrary connection on $F^*(TM)$ (not necessarily pulled back from a connection on $TM$). Consider the bilinear, antisymmetric "pseudo-torsion" map $\tilde{\tau}_\psi = \tilde{\tau}_\psi^{\nabla'} : \Gamma(TN) \times \Gamma(TN) \to \Gamma(F^*(TM))$ defined by

$$\tilde{\tau}_\psi(X,Y) = \nabla'_X\hat{Y} - \nabla'_Y\hat{X} - \widehat{[X,Y]}.$$

(The subscript $\psi$ is for "pseudo"; there is no object "$\psi$" here.)

Show that $\tilde{\tau}_\psi$ is $\mathcal{F}(N)$-bilinear, hence tensorial, defining a section $\tau_\psi = \tau_\psi^{\nabla'} \in \Omega^2(N; F^*(TM))$.

(b) We may view (3) as the definition of a canonical $F^*(TM)$-valued 1-form $I_\psi$ on $N$,

$$I_\psi(X_p) = \hat{X}_p = F_p^\sharp(F_{*p}X_p).$$

Show that $\tau_\psi = d_{\nabla'} I_\psi$.

(c) Show that the condition $\tau_\psi^{\nabla'} \equiv 0$ is equivalent to the statement that for all local-coordinate systems $\{x^i\}$ on $N$,

$$\nabla'_{\frac{\partial}{\partial x^i}} \left( \widehat{\frac{\partial}{\partial x^j}} \right) = \nabla'_{\frac{\partial}{\partial x^j}} \left( \widehat{\frac{\partial}{\partial x^i}} \right) \quad \text{for all } i, j. \tag{4}$$

(d) Show that if $\nabla'$ is the pullback of a connection $\nabla$ on $TM$ whose torsion is $\tau = \tau^\nabla$, then $\tau_\psi^{\nabla'} = F^*\tau$, where we define $F^*\tau$ pointwise by

$$(F_p^*\tau)_p(X_p, Y_p) := F_p^\sharp \left( \tau_{F(p)}(F_{*p}X_p, F_{*p}Y_p) \right), \quad p \in N.$$

*Hint*: Fix an arbitrary point $p \in N$ and let $\{x^i\}, \{y^i\}$ be local coordinates on a neighborhood of $p, F(p)$ respectively. Compute $\tilde{\tau}_\psi \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$. The Jacobian $\left( \frac{\partial y^i}{\partial x^j} \right)$ will enter your calculation.

(e) Use earlier parts of this problem to show that if $\nabla$ is any torsion-free connection on $TM$ (e.g. the Levi-Civita connection of a Riemannian metric), then (4) holds.

**Remark 2** This underlies and generalizes a crucial step in the derivation of formulas for the first and second variations of the arclength and energy functions on the space of paths between two points in $M$.