Differential Geometry III—MAT 6932/4930 —Fall 2015
Assignment 1 (possibly not complete yet)

1. Let M be a manifold. Recall that for any finite-dimensional vector space V, the canonical isomorphism $V^* \otimes V \to \mathrm{End}(V)$ is the unique linear map such that for all $\xi \in V^*, w \in V$,

$$\xi \otimes w \mapsto \{\text{the linear map } v \mapsto \langle \xi, v \rangle w\},$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing $V^* \otimes V \to \mathbf{R}$.

(a) Let $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ be a connection on TM. Define a map

$$\begin{aligned} \widetilde{\nabla} : \Gamma(TM) &\to \Gamma(T^*M \otimes TM) \underset{\text{canon.}}{\cong} \Gamma(\mathrm{End}(TM)), \\ Y &\mapsto \widetilde{\nabla} Y, \end{aligned}$$

by setting

$$\underbrace{(\widetilde{\nabla} Y)_p}_{\in \mathrm{End}(T_pM)} (X_p) = \nabla_{X_p} Y \tag{1}$$

for all $Y \in \Gamma(TM), p \in M, X_p \in T_pM$. Show that $\widetilde{\nabla}$ is Leibnizian in the following sense: for all $Y, Z \in \Gamma(TM)$ and all $f : M \to \mathbf{R}$,

$$\widetilde{\nabla}(Y + Z) = \widetilde{\nabla} Y + \widetilde{\nabla} Z \quad \text{and} \tag{2}$$
$$\widetilde{\nabla}(fY) = df \otimes Y + f \widetilde{\nabla} Y. \tag{3}$$

(b) Conversely, suppose that $\widetilde{\nabla} : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ is Leibnizian in the sense of equations (2)–(3), and use equation (1) to define a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. Show that ∇ is a connection on TM.

(c) Let $\widetilde{\nabla}$ be as above, let $p \in M$, let $\{e_i\}$ be an arbitrary basis of T_pM, and let $\{\theta^i\}$ be the basis of T_p^*M dual to $\{e_i\}$. Show that equation (1) is equivalent to

$$(\widetilde{\nabla} Y)_p = \sum_i \theta^i \otimes \nabla_{e_i} Y. \tag{4}$$

Remark. Another definition of "connection on TM" is a map of the form $\widetilde{\nabla}$ above; parts (a) and (b) above show that each of the objects ∇ and $\widetilde{\nabla}$ canonically determines the other. For this reason, we usually do not distinguish between ∇ and $\widetilde{\nabla}$ notationally. Henceforth, in the context of connections, we allow the same symbol ∇ (with no subscripts or arguments) to stand for both the map $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ and the canonically associated map $\Gamma(TM) \to \Gamma(T^*M \otimes TM)$.

2. **Definition**. Let M be a manifold and let ∇ be a connection on TM. We call a vector field Y *covariantly constant*, or *parallel*, if $\nabla Y = 0$ (equivalently, if $\nabla_X Y = 0$ for all vector fields X).

(a) Let $M = \mathbf{R}^n$, and let ∇ be the "standard flat connection" D introduced in class. Find all the covariantly constant vector fields. (You should find that these vector fields form an n-dimensional vector subspace P of the infinite-dimensional space of vector fields on $\mathbf{R}^n$, with the further property that if $\{Y_1, \ldots, Y_n\}$ is a basis of P, then for all $p \in \mathbf{R}^n$, the set $\{Y_1|_p, \ldots Y_n|_p\}$ is a basis of $T_p\mathbf{R}^n$.)

(b) Let M be any manifold, ∇ a connection on TM, and let $P := P(\nabla) \subset \Gamma(TM)$ be the set of covariantly constant vector fields. Show that P is a vector space.

Remark: For connected M, it can be shown (using a tool we don't have yet) that $\dim(P(\nabla)) \leq \dim(M)$ for every connection ∇.

(c) Let M be an n-manifold, ∇ a connection on TM, and assume that there exist n covariantly constant vector fields $Y_1, \ldots, Y_n$ whose values at each $p \in M$ form a basis of T_pM (as was the case in part (a)). Show that the connection ∇ is flat.

Remark: Every n-manifold M including $\mathbf{R}^n$, *most* connections on TM admit *no* covariantly constant vector fields other than the identically-zero vector field. If M has nonzero Euler characteristic, *no* connection admits *any* nontrivial, covariantly constant vector fields.[1] However, as the next exercise shows, $\mathbf{R}^n$ is not the only n-manifold that admits a connection ∇ with an n-dimensional space of covariantly constant vector fields.

3. (**Optional problem**.) A manifold M is called *parallelizable* if TM is a trivial vector bundle (i.e. if TM is isomorphic to the product bundle $M \times \mathbf{R}^n$, where $n = \dim(M)$. Parallelizability is equivalently to the existence of n vector fields $X_1, \ldots, X_n$ whose values at each $p \in M$ form a basis of T_pM. (If such vector fields exist, then an isomorphism from the product bundle $M \times \mathbf{R}^n$ to TM is then given by $(p, (c_1, c_2, \ldots, c_n)) \mapsto \sum_i c_i X_i(p)$.) Such a set of vector fields is called a *trivialization* of TM.[2]

(a) Let M be parallelizable and let $\{X_i\}_{i=1}^n$ be a trivialization of TM. Show that there is a (unique) connection ∇ on TM such that each X_i is covariantly constant. (In view of problem 2(c), such a connection is automatically flat.)

(b) Every Lie group G is parallelizable: any basis of the space of left-invariant vector fields (LIVFs) is a trivialization, and so is any basis of the space of right-invariant vector fields (RIVFs). Show that, correspondingly, there exist connections ∇^L, ∇^R on TG such that $\nabla^L Y = 0$ for every LIVF and $\nabla^R Y = 0$ for every RIVF.

(c) Let G be a Lie group with identity element e and Lie algebra $\mathfrak{g} = T_eG$. Show that the torsion tensors of the connections ∇^L, ∇^R in (b) satisfy

[1] Zero Euler characteristic is actually a necessary and sufficient condition for TM to admit a connection that has a nontrivial covariantly constant vector field.

[2] A bundle isomorphism $TM \to M \times \mathbf{R}^n$ is also called a trivialization of TM.

$$\tau^{\nabla^L}(X_e, Y_e) = -[X_e, Y_e], \tag{5}$$
$$\tau^{\nabla^R}(X_e, Y_e) = [X_e, Y_e], \tag{6}$$

where $X_e, Y_e \in \mathfrak{g}$ and $[\cdot,\cdot]$ in is the Lie bracket on $\mathfrak{g}$. (To get (6) after you've gotten (5), the following reminders may be helpful: (i) If X is a LIVF on G, and $\iota : G \to G$ is the inversion map, then $\iota_* X$ is a RIVF. (ii) We have $\iota_{*e} = -\mathrm{id}_{\mathfrak{g}}$.) Thus, the connections ∇^L, ∇^R on TG are flat but have nonzero torsion (unless $\mathfrak{g}$ is abelian, in which case these two connections coincide and the torsion of each is zero).

(d) Show that if M is a manifold and $\nabla^{(1)}, \nabla^{(2)}$ are connections on TM, then for any functions $f_1, f_2 : M \to \mathbf{R}$ satisfying $f_1 + f_2 = 1$ (identically), the map $f_1\nabla^{(1)} + f_2\nabla^{(2)} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by

$$(f_1\nabla^{(1)} + f_2\nabla^{(2)})_X Y := (f_1\nabla^{(1)} + f_2\nabla^{(2)})(X, Y) := f_1\nabla^{(1)}_X Y + f_2\nabla^{(2)}_X Y$$

is a connection on TM.

(e) Let G be a Lie group and let ∇^L, ∇^R be as in part (c). By part (d), the map $\nabla^0 : \Gamma(TG) \times \Gamma(TG) \to \Gamma(TG)$

$$\nabla^0 = \frac{1}{2}\nabla^L + \frac{1}{2}\nabla^R$$

is a connection on TG. Show that this connection has zero torsion at the point $e \in G$.

Remark. It can be shown that the torsion of ∇^0 is identically zero, and that the curvature of ∇^0 satisfies

$$R^{\nabla^0}(X, Y)Z = -\frac{1}{4}[[X, Y], Z] \quad \text{for all LIVFs } X, Y, Z.$$

(It will be easier to establish these facts once we've talked about *covariant differentiation along a curve*, so I am not suggesting that you try to establish them now.) Thus a Lie group with nonabelian Lie algebra has three "special" connections ∇^L, ∇^R, and ∇^0. The connections ∇^L, ∇^R have zero curvature but nonzero torsion, while the connection ∇^0 has zero torsion but nonzero curvature.