Differential Geometry III—MAT 6932/4930 —Fall 2015
Assignment 2

1. Let (M, g) be a Riemannian manifold. Let c be a positive constant and let $g_{\text{rescaled}} = c^2 g$.

(a) Show that the Levi-Civita connections (M, g) and (M, g_{rescaled}) are identical.

(b) Show that the Riemann tensors of (M, g) and (M, g_{rescaled}) are identical. (Here, "Riemann tensor" means the "$R(X, Y)Z$" version, not the "$g(R(X, Y)Z, W)$" version.)

(c) Let σ and σ_{rescaled} be the sectional-curvature functions of (M, g) and (M, g_{rescaled}). Show that $\sigma_{\text{rescaled}} = c^{-2}\sigma$.

Note: The reason for writing the rescaling factor as c^2, rather than just c, is that if g is rescaled by c^2, then distances in M—which we have not yet defined—end up being rescaled by c (which is the usual meaning of "rescaling by c" in a metric space). The sphere of radius r in $\mathbf{R}^{n+1}$ is isometric to the unit sphere S^n with metric $r^2 g_{\text{std}}$, where g_{std} is the standard metric on S^n.

2. Let M be an n-dimensional manifold and let ∇ be a connection on TM. Let $\mathbf{R}_{n\times n}$ denote the space of all $n \times n$ matrices with real entries, and recall that $GL(n, \mathbf{R})$ is the set of all invertible such matrices, an open subset of $\mathbf{R}_{n\times n}$.

Let $U \subset M$ be open, and assume that $TM|_U$ has a "basis of sections", i.e. a set $\{e_1, \ldots e_n\}$ of vector fields e_i on U whose values at each $p \in U$ are a basis of T_pM. Let $\{e'_1, \ldots, e'_n\}$ be another basis of sections of $TM|_U$. Necessarily, the second basis is related to the first basis by

$$e'_j = \sum_{i=1}^{n} e_i G^i{}_j, \quad 1 \le j \le n,$$

for a unique, smooth function $G : U \to GL(n, \mathbf{R}) \subset \mathbf{R}_{n\times n}$. (At each $p \in U$, the $G^i{}_j(p)$ are the entries of $G(p)$.)

Let Θ, Θ' be the connection forms of ∇ relative to the bases $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$, respectively.

(a) Show that

$$\Theta' = G^{-1}\Theta G + G^{-1}dG, \tag{1}$$

where G^{-1} and G are treated as matrices whose entries are real-valued functions; Θ, Θ', and dG are treated as matrices whose entries are real-valued 1-forms; and $(dG)^i{}_j = d(G^i{}_j)$. Helpful observation: (1) is equivalent to

$$\Theta'(X) = G^{-1}\Theta(X)G + G^{-1}X(G) \quad \forall X \in \Gamma(TM|_U). \tag{2}$$

In (2), all of the objects $\Theta'(X), \Theta(X), G^{-1}, G$, and $X(G)$ may be viewed either as $\mathbf{R}_{n\times n}$-valued functions, or as matrices whose entries are real-valued functions. In the former point of view, at each $p \in U$, $X_p(G)$ is the directional derivative of the $\mathbf{R}_{n\times n}$-valued function G in the direction $X_p \in T_pM$; in the latter point of view, $X_p(G)$ is a matrix whose $(i, j)^{\text{th}}$ entry is $X_p(G^i{}_j)$. In case you've forgotten (or never learned) how to compute $d(G^{-1})$ for a

$GL(n, \mathbf{R})$-valued function, the formula can be found by formally taking d of both sides of $GG^{-1} =$ constant function I.

(b) Show directly from (2) that

$$d\Theta' + \Theta' \wedge \Theta' = G^{-1}(d\Theta + \Theta \wedge \Theta)G \tag{3}$$

("Directly" means: don't use the fact that $(d\Theta + \Theta \wedge \Theta)(X, Y)$ is, pointwise, the matrix of the endomorpism $Z \mapsto R(X, Y)Z$ with respect to the basis $\{e_i\}$. What you're doing in (b) is a consistency-check on that fact.)

Observe that even though dG enters in (2) (implying that, in local coordinates, the first partial derivatives of the $G^i{}_j$ enter), there are no derivatives of G in (3); the value of the left-hand side at a point p can be computed just from $G(p)$ and $(d\Theta+\Theta\wedge\Theta)|_p \in \mathbf{R}_{n\times n}\otimes\bigwedge^2 T_p^* M$.

3. **Hyperbolic space.** Let $M = \mathbf{R}^n_+ := \{(x^1, \dots, x^n \in \mathbf{R}^n \mid x^n > 0\}$. Define a Riemannian metric g on M by

$$g = \frac{1}{(x^n)^2}\, g_{\mathrm{Euc}}, \tag{4}$$

where g_{Euc} is restriction to $\mathbf{R}^n_+$ of the standard Riemannian metric on $\mathbf{R}^n$. Let ∇ be the Levi-Civita connection on (M, g)

(a) Let Θ be the connection-form of ∇ with respect to the coordinate-basis vector fields $\{\frac{\partial}{\partial x^i}\}$, where $\{x^i\}$ are the standard coordinates on $\mathbf{R}^n$ (restricted to the open set $\mathbf{R}^n_+$. Show that

$$\Theta^i{}_j = \frac{1}{x^n}\left(-\delta_{ij}dx^n - \delta_{jn}dx^i + \delta_{in}dx^j\right).$$

(b) Let $K = d\Theta + \Theta \wedge \Theta$. Viewing K as an $n \times n$ matrix of real-valued 2-forms on $\mathbf{R}^n_+$, show that the entries $K^i{}_j$ of this matrix are given by

$$K^i{}_j = -\frac{dx^i \wedge dx^j}{(x^n)^2}\ .$$

(c) Show that (M, g_{Euc}) has constant sectional curvature -1.

Remark. There is a property we haven't defined or discussed yet, *completeness*, that a given Riemannian manifold may or may not have. Fact: up to isometry, for each $n \geq 2$ there is a unique complete, connected, simply connected Riemannian n-manifold with constant sectional curvature -1. Any such manifold is called *hyperbolic (n-)space*, or a *model* of hyperbolic n-space (since such a manifold is unique only up to isometry). The Riemannian manifold (M, g) above is called the *upper half-space model* of hyperbolic n-space. There is also a famous *unit disk model* of hyperbolic n-space, in which M' is the open unit disk in $\mathbf{R}^n$, centered at the origin, with Riemannian metric

$$g' = \frac{4}{(1 - r^2)^2}\, g_{\mathrm{Euc}}, \tag{5}$$

where r denotes distance to the origin. Observe how similar the right-hand side of (5) is to $\frac{4}{(1+r^2)^2}\, g_{\rm Euc},$, which is pullback of the standard metric on S^n under the inverse of the stereographic-projection map $S^n \setminus \{\text{north pole}\}$, hence has constant curvature $+1$.

(d) (**Optional**). Show by direct computation that (M', g') has constant sectional curvature -1.

(e) (**Optional**). Find an explicit isometry $(M, g) \to (M', g')$ or $(M', g') \to (M, g)$.

4. (**Optional**). Let $M, n, \nabla, U, \{e_i\}$, and Θ be as in problem 2. Let $\{\theta^i\}$ be the "basis of 1-forms on U dual to $\{e_i\}$", i.e. the (ordered) set of 1-forms on U for which $\{\theta^i|_p\}$ is the basis of T_p^*M dual to the basis $\{e_i|_p\}$ of T_pM for all $p \in U$.

The torsion tensor field $\tau = \tau^\nabla$ may be viewed as 2-form with values in TM, a section of $TM \otimes \bigwedge^2 T^*M$. (At each $p \in M$, τ_p is an antisymmetric bilinear map $T_pM \times T_pM \to T_pM$. We can canonically identify the space of such maps with $T_pM \otimes \bigwedge^2 T_p^*M$ or with $\bigwedge^2 T_p^*M \otimes T_pM$. We make the latter choice in this problem. This is similar to the ordering of tensor-product factors we chose in writing the curvature tensor field R^∇ as a section of $TM \otimes T^*M \otimes \bigwedge^2 T^*M \underset{\text{canon.}}{\cong} \text{End}(TM) \otimes \bigwedge^2 T^*M$.) Show that on U we have

$$\tau = e_i \otimes (d\theta^i + \Theta^i{}_j \wedge \theta^j). \tag{6}$$

Consequently, the vanishing of τ on U is equivalent to

$$d\theta^i + \Theta^i{}_j \wedge \theta^j = 0, \quad 1 \le i \le n. \tag{7}$$

If we assemble $\{\theta^i\}$ into a column vector to avail ourselves matrix-multiplication notation, we can write (7) more compactly as

$$d\theta + \Theta \wedge \theta = 0. \tag{8}$$