Differential Geometry III—MAT 6932/4930 —Fall 2015
Assignment 3

In this assignment, for any Riemannian manifold (M, g), the connection on TM that we choose is always the Levi-Civita connection.

1. Let (M, g) be a Riemannian manifold, $I \subset \mathbf{R}$ an interval, and $\gamma : I \to M$ a smooth curve for which $\gamma'(t)$ is nowhere zero. Assume that γ satisfies

$$\nabla_{\gamma'}\gamma' = f\gamma'$$

for some function $f : I \to \mathbf{R}$ (where ∇ is the Levi-Civita connection). Show that γ can be reparametrized as a geodesic. I.e. show that there exists an interval J and a diffeomorphism $\phi : J \to I$ such that $\gamma \circ \phi$ is a geodesic. (Hint: start by showing that, just as in Calculus 3, any curve with nonvanishing velocity can be reparametrized by arclength.)

2. (a) Let N be a manifold, M a manifold diffeomorphic to N, and $F : M \to N$ a diffeomorphism. Let ∇^N be a connection on TN. Since F is a diffeomorphism, every vector field X on M pushes forward to a well-defined vector field F_*X on N. Similarly, under the inverse diffeomorphism F^{-1}, every vector field Z on N pushes forward to a well-defined vector field $(F^{-1})_*Z$ on M. Recall that for a diffeomorphism, the map $F_* : \Gamma(TM) \to \Gamma(TN)$ satisfies $(F_*)^{-1} = (F^{-1})_*$, so $(F^{-1})_*Z$ is the same as $(F_*)^{-1}Z =: F^*Z$, the pullback of Z to M by F.

Define a map

$$\begin{aligned} \nabla^M : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM), \\ (X, Y) &\mapsto \nabla^M_X Y, \end{aligned}$$

by

$$\nabla^M_X Y = F_*^{-1}\left(\nabla^N_{F_*X} F_*Y\right). \tag{1}$$

Show that ∇^M is a connection on TM.

(b) Notation as in (a), but now assume that g_N is a Riemannian metric on N and that ∇^N is the Levi-Civita connection of (N, g_N). Show that ∇^M is the Levi-Civita connection of (M, F^*g_N).

(c) Let (M, g_M), (N, g_N) be Riemannian manifolds, and assume that $F : M \to N$ is an isometry (i.e. a diffeomorphism such that $F^*g_N = g_M$). Show that if γ is a geodesic in M, then $F \circ \gamma$ is a geodesic in N.[1]

[1]For a Riemannian manifold "geodesic" means "geodesic for the Levi-Civita connection" unless otherwise specified.

3. **Normal Coordinates.** Let (M^n, g) be a Riemannian manifold and let $p \in M$. A *normal neighborhood of* p is the image, under $\exp_p$, of a ball $B_\epsilon(0) \subset T_pM$, where ϵ (the *radius* of the normal neighborhood) is small enough that $\exp_p\big|_{B_\epsilon(0)}$ is a diffeomorphism onto its image.

Let U be a normal neighborhood of p of radius ϵ. Let $\mathbf{e} = \{e_i\}_1^n$ be an orthonormal basis of T_pM. Define a diffeomorphism

$$\begin{aligned} \phi_{\mathbf{e}} : (B_\epsilon(0) \subset \mathbf{R}^n) &\to U, \\ (a^1, \dots, a^n) &\mapsto \exp_p(a^i e_i) \end{aligned}$$

(Here $B_\epsilon(0) \subset \mathbf{R}^n$ is the Euclidean ϵ-ball.) Then $(U, \phi_{\mathbf{e}}^{-1})$ is a coordinate chart, and the corresponding coordinate functions x^i are called (a system of) *normal coordinates on* U, *centered at* p.

(a) Let $\{x^i\}$ be a normal-coordinate system centered at p determined by an orthonormal basis $\mathbf{e} = \{e_i\}_1^n$ of T_pM. Show that

$$\left.\frac{\partial}{\partial x^i}\right|_p = e_i \ , \quad 1 \le i \le n. \tag{2}$$

(b) Using the fact that straight lines through the origin in $B_\epsilon(0) \subset T_pM$ are mapped by $\exp_p$ to geodesics, show that

$$\left.\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)\right|_p = 0. \tag{3}$$

(Hence all the Christoffel symbols in this coordinate system vanish *at* p. In general they do not all vanish *except* at p.)

(c) Let U be a normal neighborhood of p, and let $\{x^i\}$, $\{y^i\}$ be two systems of normal coordinates on U centered at p. Show that there exists a constant orthogonal matrix A relating the two coordinate systems (i.e. $y^i = A^i{}_j x^j$).

4. **(Optional)**. **Fun with the hyperbolic plane.** Assignment 2 introduced the upper half-space model of hyperbolic n-space. The $n = 2$ case is call the *upper half-plane model of the hyperbolic plane.* For this case, let x and y denote the standard coordinates on $\mathbf{R}^2$, so that $\mathbf{R}^2_+ = \{(x, y) \in \mathbf{R}^2 \mid y > 0\}$ and the Euclidean metric is $g_{\text{Euc}} = dx^2 + dy^2 := dx \otimes dx + dy \otimes dy$. In this notation, the hyperbolic metric on $\mathbf{R}^2_+$ is

$$g = g_{\text{hyp}} = \frac{dx^2 + dy^2}{y^2} \ .$$

From Assignment 2, we know that $(\mathbf{R}^2_+, g_{\text{hyp}})$ has constant sectional curvature -1. Below we discover some other interesting features of the hyperbolic plane (as viewed through the upper half-plane model).

(a) Let $x_0 \in \mathbf{R}$, and let C be an open semicircle in the upper half-plane centered at $(x_0, 0)$ (i.e. $\{(x,y) \in \mathcal{H}^2 \mid (x-x_0)^2 + y^2 = R^2\}$ for some $R > 0$). Choose a parametrization γ of C. Show that γ can be reparametrized as a geodesic.

(b) Same as part (a), but for the vertical ray $C = \{(x_0, y) \mid y > 0\}$.

Remarks. (1) It is easy to see that given a point p in the upper half-plane, and a non-vertical straight line ℓ through (x_1, y_1), there exists a unique circle centered on the x-axis that is tangent to ℓ at p. It follows that the image of every geodesic in $(\mathcal{H}^2, g_{\rm hyp})$ has image lying in one of the semicircles or vertical rays considered above.

(2) An alternate way of obtaining the results in part (a) and (b) is as follows. Step 1: Do part (c) below. Step 2: Show that the y-axis, suitably parametrized, is the image of a unit-speed geodesic γ_0 with domain $(-\infty, \infty)$ and $\gamma_0(0) = (0, 1)$. Step 3: Using problem 2, show that every unit-speed geodesic, parametrized over its maximal domain, is $f_A \circ \gamma_0$ (see part (b)) for some $A \in SL(2, \mathbf{R})$. Step 4: Show that as A ranges over $SL(2, \mathbf{R})$, the images of the y-axis are precisely the semicircles and vertical lines in (a) and (b).

(c) Identify $(x, y) \in \mathbf{R}^2$ with the complex number $z = x + yi$; this identifies $\mathbf{R}^2_+$ with $H := \{z \in \mathbf{C} \mid \operatorname{im}(z) > 0\}$. Recall that $SL(2, \mathbf{R})$ denotes the group of real 2×2 matrices of determinant 1. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ define the *linear fractional transformation* $f_A : H \to \mathbf{C}$ by

$$f_A(z) = \frac{az + b}{cz + d}$$

(note that the condition $\operatorname{im}(z) > 0$ ensures that $cz + d \neq 0$).

(i) Show that for all $A \in SL(2, \mathbf{R})$, we have $f_A(H) = H$. Thus f_A is a diffeomorphism $\mathbf{R}^2_+ \to \mathbf{R}^2_+$. Furthermore, f_A preserves orientation (this is a consequence of the fact that, viewed as a map $H \subset \mathbf{C} \to \mathbf{C}$, is holomorphic).

(ii) Show that for $A, B \in SL(2, \mathbf{R})$, $f_{AB} = f_A \circ f_B$. Thus the map $SL(2, \mathbf{R}) \times \mathbf{R}^2_+ \to \mathbf{R}^2_+$, $(A, z) \mapsto A \cdot z := f_A(z)$, is a left-action of $SL(2, \mathbf{R})$ on $\mathbf{R}^2_+$.

(iii) Show that for each $A \in SL(2, \mathbf{R})$, the map $f_A : \mathbf{R}^2_+ \to \mathbf{R}^2_+$ preserves the hyperbolic metric:

$$(f_A)^* g_{\rm hyp} = g_{\rm hyp} \ .$$

Remark: Thus the action of $SL(2, \mathbf{R})$ on the hyperbolic plane is an action by orientation-preserving isometries. Writing $\operatorname{Isom}_+(M, g)$ for the group of isometries of an orientable, connected Riemannian manifold (M, g), the map $A \mapsto f_A$ is a homomorphism $SL(2, \mathbf{R}) \to \operatorname{Isom}_+(\mathbf{R}^2_+, g_{\rm hyp})$. This homomorphism has a nontrivial kernel, the $\mathbf{Z}_2$-subgroup $\{\pm I\}$ (which happens to be the center of $SL(2,\mathbf{R})$[2]). Thus the quotient group $PSL(2,\mathbf{R}) := SL(2,\mathbf{R})/(\text{center})$ acts faithfully as a group of orientation-preserving isometries of the hyperbolic plane. It can be shown that there are no other orientation-preserving isometries: the map $A \mapsto f_A$ is a surjection from $SL(2,\mathbf{R})$ to $\text{Isom}_+(\mathbf{R}^2_+, g_{\text{hyp}})$. Thus $PSL(2,\mathbf{R})$, identified with the group of diffeomorphisms given by the $SL(2,\mathbf{R})$-action, is the full group $\text{Isom}_+(\mathbf{R}^2_+, g_{\text{hyp}})$.

[2]The *center* of a group G is the subgroup $\{h \in G \mid gh = hg \quad \forall g \in G\}$.