Differential Geometry III—MAT 6932/4930 —Fall 2015
Assignment 4

In this assignment, for any Riemannian manifold (M, g), unless otherwise specified the connection ∇ on TM always denotes the Levi-Civita connection, and n always denotes $\dim(M)$.

1. **"Decoupling" of the Jacobi Equation.** Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ a non-constant geodesic (I any interval), and let J be a Jacobi field along γ. Let the vector fields $J^{\rm par}$ and $J^{\perp}$ along γ be the tangential and normal components of J; i.e.

$$J^{\rm par} = g(J, \hat{\mathbf{T}})\hat{\mathbf{T}}, \qquad J^{\perp} = J - J^{\rm par},$$

where $\hat{\mathbf{T}}(t) = \mathbf{T}(t)/\|\mathbf{T}(t)\|$ and $\mathbf{T}(t) = \gamma'(t)$ for all $t \in I$.

(a) Show that $J^{\rm par}$ and $J^{\perp}$ are Jacobi fields along γ.

For the rest of this problem, call the Jacobi field J *tangential* if $J = f\mathbf{T}$ for some function $f : I \to \mathbf{R}$, and *normal* if $J(t) \perp \mathbf{T}(t)$ for all $t \in I$.

(b) Show that if J is tangential, then for some $a, b \in \mathbf{R}$ and all $t \in I$ we have $J(t) = (at + b)\mathbf{T}(t)$.

(c) Show that J is tangential if and only if both $J(0)$ and $(\nabla_T J)(0)$ are proportional to $\mathbf{T}(0)$.

(d) Show that J is normal if and only if both $J(0)$ and $(\nabla_T J)(0)$ are perpendicular to $\mathbf{T}(0)$.

2. **Covariant derivative of certain tensor fields**. Let M be a manifold and let ∇ be an arbitrary connection on TM. The connection ∇ determines a way of covariantly differentiating any tensor field on M. In this problem, for the sake of brevity, we will discuss only how to covariantly differentiate sections of the bundle $E = T^*M \otimes T^*M \otimes T^*M \otimes TM$. (Our motivation for this choice is that the Riemann tensor [field] of a Riemannian metric is a section of this bundle. In a later problem, we will want to covariantly differentiate the Riemann tensor.) The value at $p \in M$ of a "set-theoretic" section S of this bundle is a trilinear map $S_p : T_pM \times T_pM \times T_pM \to T_pM$. For vector fields X, Y, Z, the vector field $S(X, Y, Z)$ is defined pointwise by

$$S(X, Y, Z)|_p = S_p(X_p, Y_p, Z_p).$$

If $S(X, Y, Z)$ is smooth for all vector fields X, Y, Z, then we drop the words "set-theoretic" and call S a section (= smooth section) of E, and write $S \in \Gamma(E)$.

Let $S \in \Gamma(E)$. For vector fields X, Y, Z, W on M, define

$$(\widetilde{\nabla}_W S)(X,Y,Z) = \nabla_W\left(S(X,Y,Z)\right) - S(\nabla_W X, Y, Z) - S(X, \nabla_W Y, Z) - S(X, Y, \nabla_W Z), \quad (1)$$

a vector field on M.

(a) Show that $(\widetilde{\nabla}_W S)(X,Y,Z)$ is $\mathcal{F}$-quadrilinear in the variables X, Y, Z, W.

As a consequence of $\mathcal{F}$-multilinearity, with S held fixed, for each $p \in M$ the value of $(\widetilde{\nabla}_W S)(X,Y,Z)$ at p depends only on the values of X, Y, Z, W at p. Thus, given only $X_p, Y_p, Z_p, W_p \in T_pM$, we can (and do) unambiguously define a vector $(\nabla_W S)_p(X_p, Y_p, Z_p) \in T_pM$ by setting $(\nabla_W S)_p(X_p, Y_p, Z_p) = \left((\widetilde{\nabla}_W S)(X,Y,Z)\right)\Big|_p$, where X, Y, Z, W are arbitrary smooth extensions of X_p, Y_p, Z_p, W_p. In view of this fact, we henceforth write $\widetilde{\nabla}_W S$ just as $\nabla_W S$, and interpret "$\nabla_W S$" as a section of E if W is a vector field, or as an element of the fiber $E_p = T_p^*M \otimes T_p^*M \otimes T_p^*M \otimes T_pM$ if W is just a single tangent vector at a point p.

Observe that (1) can now be written as

$$\nabla_W\left(S(X,Y,Z)\right) = (\nabla_W S)(X,Y,Z) + S(\nabla_W X, Y, Z) + S(X, \nabla_W Y, Z) + S(X, Y, \nabla_W Z). \quad (2)$$

But note that on the right-hand side of (2), the first "∇_W" is differentiating a section of E, while the other three are differentiating sections of TM.

(b) Show that, for a fixed vector field W, the expression $\nabla_W S$ is Leibnizian in S (i.e. additive in S and satisfying

$$\nabla_W(fS) = W(f)S + f\nabla_W S$$

for all smooth $f : M \to \mathbf{R}$).

Because the map $(W, S) \mapsto \nabla_W S$ is $\mathcal{F}$-linear in W and Leibnizian in S, and is constructed canonically from the connection ∇ on TM, we call this map the *connection on E induced by the connection ∇ on TM.*

Remark. If R is the Riemann tensor field of some Riemannian metric, recall that R is a section of the bundle E above, and it is simply our notational choice to write "$R(X,Y)Z$" instead of "$R(X,Y,Z)$". Thus, for a Riemann tensor, and a connection ∇ on TM (which, in practice, will usually be the Levi-Civita connection), equation (2) is written as

$$\nabla_W\left(R(X,Y)Z\right) = (\nabla_W R)(X,Y)Z + R(\nabla_W X, Y)Z + R(X, \nabla_W Y)Z + R(X,Y)\nabla_W Z. \quad (3)$$

3. Taylor expansion of the metric in normal coordinates.

See Assignment 3, problem 3, for terminology used in this problem. You are allowed to use results of that problem in the current problem.

Let (M^n, g) be a Riemannian manifold, let $p \in M$, let U be a normal neighborhood of p, and let $B = B_\delta(0) \subset T_pM$ be the ball for which $\exp_p\big|_B : B \to U$ is a diffeomorphism. Let $\mathbf{e} = \{e_i\}_1^n$ be an orthonormal basis of T_pM, and let $\{x^i\}_{i=1}^n$ be the associated system of normal coordinates on U.

With a slight abuse of notation, let $x = x^i e_i$ denote a (temporarily) fixed but arbitrary point of B, so that $\exp_p(x)$ is exactly the point whose normal coordinates are $(x^1, \dots, x^n)$. (Alternatively, temporarily fix $q \in U$. We are simply abbreviating the normal coordinates $x^i(q)$ as x^i to simplify some formulas below.) Let $w = w^i e_i \in T_pM$. As discussed in class, for $\epsilon > 0$ sufficiently small, we can define a smooth map $\alpha : (-\epsilon, \epsilon) \times [0,1] \to M$ by $\alpha(s,t) = \exp_p(t(x + sw))$. For $|s| < \epsilon$ we define $\bar{\alpha}_s : [0,1] \to M$ by $\bar{\alpha}_s(t) = \alpha(s,t)$. Let $\gamma := \bar{\alpha}_0$. Then $\bar{\alpha}$ is a variation of γ through geodesics, so the variation vector field $J := \frac{d\bar{\alpha}}{ds}\big|_{s=0} := \frac{\partial \alpha}{\partial s}\big|_{s=0}$ is a Jacobi field along γ. Let $T = \gamma'$, and for any vector field Y along γ, let $Y' = \nabla_T Y, Y'' = \nabla_T \nabla_T Y$, etc.

(a) Show that $J(0) = 0$, $J'(0) = w$, and $J(1) = w^i \frac{\partial}{\partial x^i}\big|_{\exp_p(x)}$. (We saw the first two of these equalities in class, but the argument for the second was only sketched verbally. We also saw the third equality, just written differently.)

(b) Let $f(t) = \|J(t)\|^2$, so that

$$
\begin{aligned}
f'(t) &= 2g(J', J), \\
f''(t) &= 2g(J'', J) + 2g(J', J'), \\
f'''(t) &= 2g(J''', J) + 6g(J'', J'), \\
f^{(iv)}(t) &= 2g(J^{(iv)}, J) + 8g(J''', J') + 6(J'', J''),
\end{aligned}
$$

etc. Using the Jacobi equation $J'' = R(T, J)T$, show that the m^{th} derivative of f can be computed as a universal expression in g, T, J, J', the Riemann tensor R, and the covariant derivatives $(\nabla_T)^i R$ up to order $m-2$. (Your work should lead you to the understanding of the phrase "universal expression" in this context, but I am not asking you to give a formal definition of what it means. Do not try to derive a formula for the universal expression for arbitrary m; the point of this exercise is only to show that you *could*, in principle, find such a formula for any desired order m.)

(c) With f as above, show that $f(0) = 0 = f'(0) = f'''(0)$, that $f''(0) = 2\|w\|^2$, and that $f^{(iv)}(0) = 8g(R(T,w)T, w)|_0 = -8R_{ikjl}w^i w^j x^k x^l$, where $\{R_{ikjl}\}$ are the components of the (four-lower-index) Riemann tensor at p in the tensor-space basis determined by the basis $\{e_i\}$ of T_pM. (There is no misprint in the order of indices above; the w's are paired with the first and third indices of R, and the x's are paired with the second and fourth.) Hence show that

$$\begin{aligned} f(t) &= t^2\|w\|^2 - \frac{1}{3}t^4 R_{ikjl}w^i w^j x^k x^l + O(t^5\|w\|^2) \\ &= t^2 w^i w^j \left(\delta_{ij} - \frac{1}{3}t^2 R_{ikjl}x^k x^l + O(t^3)\right). \end{aligned}$$

(d) Using parts (a) and (c) above, show that the metric components $g_{ij}(x) := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\big|_{\exp_p(x)}$ satisfy

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l + O(|x|^3), \tag{4}$$

where $|x| = \left(\sum_i (x^i)^2\right)^{1/2}$.

(e) Show that if the radius of B is chosen small enough, or if (M, g) is complete, then $|x| = \rho(p, \exp_p(x))$, where ρ is the distance-function on M determined by g. (This part of the problem is completely independent of the preceding parts. Its purpose is just to provide a geometric interpretation of the term "$O(|x|^3)$" in (4).) *Note*: This requirement on radius of B is part of many authors' definitions of "normal neighborhood". I omit it from mine because it is not necessary for the derivation of equation (4).

4. (Optional problem). Lemmas for use in later problems. Let $\{y^i\}$ be standard coordinates on $\mathbf{R}^n$, let $\omega \in \Omega^{n-1}(S^{n-1})$ be the standard volume form, and let $\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} \omega$ (the volume of the standard, Euclidean, unit sphere).

(a) Show that for all $i, j \in \{1, \ldots, n\}$,

$$\int_{S^{n-1}} y^i y^j \ \omega = \frac{1}{n}\, \delta_{ij}\, \mathrm{Vol}(S^{n-1}).$$

(There is a way to do this that does not involve any trigonometric integrals.)

(b) *If you were in my class last year, you have already done this problem, and do not need to do it again.* Show that on the complement of the origin in $\mathbf{R}^n$,

$$dy^1 \wedge \cdots \wedge dy^n = r^{n-1} dr \wedge \tilde{\omega},$$

where $\tilde{\omega} = \pi^*\omega$ is the pullback of ω via the radial projection $\pi : \mathbf{R}^n \setminus \{0\} \to S^{n-1}$, $y \mapsto y/\|y\|$.

5. Ricci tensor and scalar curvature. Let (M, g) be a Riemannian manifold. For each $p \in M$ and $X, Y \in T_pM$, the Riemann tensor defines a linear map $T_pM \to T_pM$ by $Z \mapsto R(X, Z)Y$. Define

$$\mathsf{Ric}(X, Y) = \mathsf{Ric}|_p(X, Y) = \mathrm{tr}(Z \mapsto R(X, Z)Y),$$

where "tr" denotes the trace. Thus, if $\{e_i\}$ is an arbitrary basis of T_pM and $\{\theta^i\}$ is the dual basis of T_p^*M,

$$\mathsf{Ric}(X,Y) = \langle \theta^i, R(X,e_i)Y\rangle.$$

Clearly the map $(X,Y) \mapsto \mathsf{Ric}|_p(X,Y)$ is bilinear, so $\mathsf{Ric}|_p$ is an element of $T_p^*M \otimes T_p^*M$. This tensor is called the *Ricci tensor* at p. Letting p vary, it is easily seen that $\mathsf{Ric}|_p$ depends smoothly on p, so Ric becomes a tensor field on M, called the *Ricci tensor* (field) or the *Ricci curvature.*

(a) Show that with $p, \{e_i\}, \{\theta^i\}$ as above, the Ricci tensor at p is given by

$$\begin{aligned} \mathsf{Ric} &= R_{jl}\, \theta^j \otimes \theta^l, \\ &\quad \text{where } R_{jl} = R^i{}_{jil} \end{aligned}$$

and where $\{R^i{}_{jkl}\}$ are the components of the Riemann tensor at p with respect to the given bases.

(b) Show that the Ricci tensor is a *symmetric* tensor field: for all $p \in M$ and all $X, Y \in T_pM$, we have $\mathsf{Ric}(X,Y) = \mathsf{Ric}(Y,X)$.

(c) **(Optional).** Recall that for any finite-dimensional vector space V any symmetric bilinear form $h : V \times V \to \mathbf{R}$ is determined by its restriction to the diagonal: if we know $h(X,X)$ for *all* $X \in V$, then we know $h(X,Y)$ for all $X,Y \in V$. This follows from the *polarization identity*

$$h(X,Y) = \frac{h(X+Y,X+Y) - h(X-Y,X-Y)}{4}\ .$$

Furthermore, if V is given a norm $\|\ \|$, then for all nonzero $X \in V$ we have $h(X,X) = \|X\|^2 h(\hat{X},\hat{X})$, where $\hat{X} = X/\|X\|$. Thus, in the presence of a norm, a symmetric bilinear form from the function f_h (notation just for this problem) that it determines on the unit sphere:

$$\begin{aligned} f_h : S(V) := \{X \in V : \|X\| = 1\} &\to \mathbf{R}, \\ X &\mapsto f_h(X) := h(X,X). \end{aligned}$$

In particular, for each $p \in M$, the function $f_{\mathsf{Ric}} : S(T_pM) \subset T_pM$ carries all the information of the Ricci tensor at p.

Recall that, at each p, the sectional curvature of M at p is a map $G_2(T_pM) \to \mathbf{R}$, $\mathcal{P} \mapsto \sigma(\mathcal{P})$. For $X \in S(T_pM)$ let $X^\perp = \{Y \in T_pM : Y \perp X\}$. Let $G_2^X(T_pM) \subset G_2(T_pM)$ denote the set of all 2-planes in T_pM that contain X. There is a two-to-one map

$$\begin{aligned} \pi_X : S(X^\perp) &\to G_2^X(T_pM), \\ \pi_X(Y) &= \mathcal{P}(X,Y) := \mathrm{span}\{X,Y\}. \end{aligned}$$

The vector space $X^\perp$ is a Riemannian manifold with the standard Riemannian metric determined by $g_p|_{X^\perp}$; thus $S(X^\perp)$ inherits a Riemannian metric. Orienting $X^\perp$ arbitrarily, and giving S^{n-1} the induced orientation, we then obtain a volume form form ω_{n-2} on $S(X^\perp)$. (The subscript here is just a reminder of the dimension of $S(X^\perp)$.) Show that for $X \in S(T_pM)$,

$$\int_{S(X^\perp)} (\sigma\circ\pi_X)\,\omega_{n-2} = \int_{S(X^\perp)} \sigma(\mathcal{P}(X,\cdot))\,\omega_{n-2} = \frac{\mathrm{Vol}(S^{n-2})}{n-1}\mathsf{Ric}(X,X) = \frac{\mathrm{Vol}(S(X^\perp))}{n-1} f_{\mathsf{Ric}}(X). \tag{5}$$

Remark. From (5), we may view the expression

$$\frac{1}{n-1} f_{\mathsf{Ric}}(X) = \frac{1}{\mathrm{Vol}(S(X^\perp))}\int_{S(X^\perp)} (\sigma\circ\pi_X)\,\omega_{n-2} \tag{6}$$

as representing the *average sectional curvature among all two-planes in T_pM that contain X.*[1]

(d) Let $\mathsf{g}_p : T_pM \to T_p^*M$ be the isomorphism induced by the inner product g_p. For any tensor $h_p \in T_p^*M \otimes T_p^*M$, we define the *trace of h_p with respect to g_p*, denoted $\mathrm{tr}_{g_p}(h_p)$, to be the image of h_p under the following composition maps

$$T_p^*M \otimes T_p^*M \overset{\mathsf{g}_p{}^{-1}\otimes \mathrm{id.}}{\longrightarrow} \underset{\text{canon.}}{\cong} \mathrm{Hom}(T_pM, T_pM) \overset{\text{trace}}{\longrightarrow} \mathbf{R}.$$

Applying this pointwise to any $h \in \Gamma(\mathrm{Sym}^2(T^*M))$ gives a real-valued function $\mathrm{tr}_g(h) : M \to \mathbf{R}$.

---

[1]The reason we integrated over $S(X^\perp)$ in (5) and (6), rather than over $G_2^X(T_pM)$, is that $G_2^X(T_pM)$ is diffeomorphic to the projective space $\mathbf{RP}^{n-2}$, which is not orientable when n is even. However, whether or not a Riemannian manifold (N, g_N) is orientable, the metric g_N induces a well-defined measure "$d\mu_N$" on N; it's simply something that we did not discuss last year. Therefore for any finite-dimensional inner-product space W, the projectization $\mathbf{P}(W)$ has a Riemannian metric, hence Riemannian measure $d\mu$, induced the by the natural two-to-one covering map $\pi' : S(W) \to \mathbf{P}(W)$ and the standard Riemannian metric on $S(W)$. (Here we regard W as a Riemannian manifold with the standard Riemannian metric determined by the given inner product on W.) Using these facts it can be shown $\mathrm{Vol}(S(X^\perp)) = 2\mathrm{Vol}(G_2^X(T_pM))$ and that

$$\int_{S(X^\perp)} (\sigma\circ\pi_X)\,\omega = \int_{G_2^X(T_pM)} \sigma\; d\mu.$$

Thus (6) indeed represents the average value of the function $\sigma|_{G_2^X(T_pM)}$ with respect to the induced measure on $G_2^X(T_pM)$.

Show that for h as above, $p \in M$, $\{e_i\}$ any basis of T_pM, $g_{\cdot\cdot}$ the matrix of g_p with respect to this basis, and $g^{\cdot\cdot} = (g_{\cdot\cdot})^{-1}$,

$$\mathrm{tr}_g(h)\big|_p = g^{ij}h_{ij} = h^i{}_i = h_i{}^i \ ,$$

where $h_{ij} = h(e_i, e_j)$.

(e) **(Optional, except for reading the definition of *scalar curvature*).** The *scalar curvature* or *Ricci scalar* is the real-valued function $\mathsf{R} = \mathrm{tr}_g(\mathsf{Ric})$. Show that at each $p \in M$,

$$\mathsf{R}(p) = \frac{n}{\mathrm{Vol}(S^{n-1})} \int_{S(T_pM)} f_{\mathsf{Ric}} \ \omega_{n-1}$$

where ω_{n-1} is the volume form on the sphere $S(T_pM)$ induced by the metric g_p and an arbitrary choice of orientation of T_pM.

Thus, up to a normalization constant, the scalar curvature at p is the average value of the function $S(T_pM) \to \mathbf{R}, X \mapsto f_{\mathsf{Ric}}(X)$. But $f_{\mathsf{Ric}}(X)$ is (for each X) an average of sectional curvatures, so scalar curvature is sometimes thought of as a "double average" of sectional curvatures. However, the word "double" can be eliminated: it can be shown that, up to a normalization constant $\mathsf{R}(p)$ is simply the average value of the sectional curvature σ_p over the Grassmannian $G_2(T_pM)$.

6. Riemannian volume form in arbitrary local coordinates. *If you were in my class last year, you have already done this problem, and do not need to do it again.*

Let $\{x^i\}$ be local coordinates (not necessarily normal coordinates) on an open set $U \subset M$, let $\{e_i\}$ be a pointwise-orthonormal basis of $TM|_U$, and let $\{\theta^i\}$ be the (pointwise) dual basis of sections of T^*M. Let $A : U \to GL(n, \mathbf{R})$ be the matrix-valued function relating the bases $\{dx^i\}$, $\{\theta^i\}$ by the equation $dx^i = A^i{}_j\theta^j$.

(a) Express $dx^1 \wedge \cdots \wedge dx^n$ in terms of A and $\theta^1 \wedge \cdots \wedge \theta^n$.

(b) Express the matrix $g_{\cdot\cdot}$ of metric coefficients $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ in terms of the matrix A.

(c) Assume that M is oriented and that $\{x^i\}$ is a positively oriented coordinate system. Show that Riemannian volume form ω_g can be expressed in these local coordinates by

$$\omega_g = \sqrt{\det(g_{\cdot\cdot})} \ dx^1 \wedge \cdots \wedge dx^n.$$

7. Riemannian volume form in normal coordinates. Assume (M, g) is oriented and let ω_g be the Riemannian volume form. Let $p \in M$, and let $\{e_i\}$ be a positively oriented basis of T_pM, and let $\{x^i\}$ be the corresponding system of normal coordinates centered at p.

(a) Use the results of earlier problems to show that

$$\omega_g = \left(1 - \frac{1}{6}R_{kl}\, x^k x^l + O(|x|^3)\right)\, dx^1 \wedge \cdots \wedge dx^n,$$

where $\{R_{ij}\}$ are the components of the Ricci tensor at p in the given coordinate system.

(b) **(Optional).** For $r > 0$, let $S_r^{n-1}(p) \subset M$ denote the sphere of radius r centered at p (with respect to the distance function ρ determined by g). Below, we assume r is taken small enough that $S_r^{n-1}(p)$ is the diffeomorphic image under $\exp_p$ of the sphere $\{x^i e_i : |x| = r\} \subset T_pM$ (see problem 3e). Show as $r \to 0$, the volume of this sphere is related to the volume of the Euclidean sphere of the same radius, S_r^{n-1}, by

$$\mathrm{Vol}(S_r^{n-1}(p)) = \mathrm{Vol}(S_r^{n-1})\left(1 - \frac{1}{6n}\mathsf{R}(p)r^2 + O(r^3)\right). \tag{7}$$

This quantifies (asymptotically) the statement that "larger curvature means smaller spheres", and shows that the *scalar* curvature provides the dominant correction to the Euclidean formula for the volume of sphere of a given radius. It also shows that scalar curvature can alternatively be defined by

$$\mathsf{R}(p) = 6n \lim_{r\to 0} \frac{\mathrm{Vol}(S_r^{n-1}) - \mathrm{Vol}(S_r^{n-1}(p))}{r^2} \tag{8}$$

(c) **(Optional).** With r as in part (b), let $\overline{B}_r(p) \subset M$ denote the closed ball of radius r centered at p. (For purposes of this problem, it does not matter whether we use open balls or closed balls. I've simply used closed balls because that's usually what we're thinking of when we're talking about volumes of balls.) Derive the asymptotic expansion of $\mathrm{Vol}(B_r(p))$ analogous to (7), and the analog of (8) giving $\mathsf{R}(p)$ in terms of volumes of balls.