Differential Geometry IV—MAT 6932/4930 —Spring 2016
Assignment 1

1. Let M be a manifold, E a vector bundle over M, F a sub-bundle of E, and h a Riemannian metric on E. For all $p \in M$ define $F_p^\perp \subset E_p$ to be the h_p-orthogonal complement of F_p in E_p. Let $F^\perp = \bigcup_{p\in M} F_p^\perp$.

(a) Prove that $F^\perp$ is a vector bundle over M. (As in class, "is" in such a statement means "canonically inherits the structure of".)

(b) Prove that $F^\perp \cong E/F$ (where "$\cong$" is the relation "isomorphic as vector bundles" via isomorphisms that cover the identity map id_M.)

2. For any vector space V and integer $m > 0$, let $V^{\otimes m}$ denote the m-fold tensor product $V \otimes V \otimes \ldots \otimes V$. The natural left-action of the symmetric group S_m on $V \times V \times \cdots \times V$ is multilinear, hence descends to an action on $V^{\otimes m}$. Denoting this action $S_m \times V^{\otimes m} \to V^{\otimes m}$ by $(\sigma, T) \mapsto \sigma \cdot T$, recall that a tensor $T \in V^{\otimes m}$ is called *symmetric* if $\sigma \cdot T = T$ for all $\sigma \in S_m$, and *alternating* or *(totally) antisymmetric* if $\sigma \cdot T = \mathrm{sgn}(\sigma) T$ for all $\sigma \in S_m$. Recall that the set of symmetric (respectively, antisymmetric) elements of $V^{\otimes m}$ is denoted $\mathrm{Sym}^m(V)$ (resp. $\bigwedge^m(V)$). Recall also that $\mathrm{Sym}^m(V)$ and $\bigwedge^m(V)$ are (vector) subspaces of $V^{\otimes m}$.

In class, for any two vector bundles E, F over M, we defined the tensor product $E \otimes F$, and proved in class (modulo minor steps left to the student, not to be handed in) that $E \otimes F$ is a vector bundle over M such that for all $p \in M$, $(E \otimes F)_p$ is canonically identified with $E_p \otimes F_p$. By induction, for any $m > 0$ the tensor-product-of-vector-bundles construction yields a vector bundle $E^{\otimes m} = E \otimes E \otimes \ldots \otimes E$ whose fiber at any $p \in M$ is canonically identified with $E_p^{\otimes m}$. Define $\mathrm{Sym}^m(E) = \bigcup_p \mathrm{Sym}^m(E_p)$, $\bigwedge^m(E) = \bigcup_p \bigwedge^m(E_p)$.

(a) Prove that for each $m > 0$, $\mathrm{Sym}^m(E)$ and $\bigwedge^m(E)$ are sub-bundles of $E^{\otimes m}$.

(b) Exhibit a canonical isomorphism $E \otimes E \to \mathrm{Sym}^2(E) \oplus \bigwedge^2(E)$.

(For dimensional reasons, the isomorphism in (b) cannot exist for higher-degree tensors and $\mathrm{rank}(k) > 1$: if V is a k-dimensional vector space, $1 < k < \infty$, and $m > 2$, then $\dim(V^{\otimes m}) > \dim(\mathrm{Sym}^m(V)) + \dim(\bigwedge^m(V))$.)