Differential Geometry IV—MAT 6932/4930 —Spring 2016
Assignment 2

1. Let E_1, E_2 be vector bundles over a manifold M, with connections $\nabla^{E_1}, \nabla^{E_2}$ respectively. A section of $E_1 \oplus E_2$ can be written as an ordered pair (s_1, s_2), where $s_i \in \Gamma(E_i)$. For later parts of this problem, it is best to write this ordered pair as a 2-component column vector. Define $\nabla = \nabla^{E_1 \oplus E_2} : \Gamma(E_1 \oplus E_2) \to \Gamma(E_1 \oplus E_2) \otimes T^*M)$ by

$$\nabla_X \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \nabla_X^{E_1} s_1 \\ \nabla_X^{E_2} s_2 \end{pmatrix} \quad \forall X \in \Gamma(TM).$$

(a) Show that ∇ is a connection on $E_1 \oplus E_2$. We call ∇ the *(induced) direct-sum connection.*

(b) Let $\Theta^{E_1}, \Theta^{E_2}$ be the connection forms of $\nabla^{E_1}, \nabla^{E_2}$ with respect to some local trivializations of E_1, E_2 over a common open set U. Introducing an appropriately induced local trivialization of $E_1 \oplus E_2$ over U, express the corresponding connection form $\Theta = \Theta^{E_1 \oplus E_2}$ of ∇ in terms of Θ^{E_1} and Θ^{E_2}.

(c) For $i = 1, 2$ let $F^{E_i} \in \Omega^2(\text{End}(E_i))$ denote the curvature of ∇^{E_i}. Let $F_\nabla \in \Omega^2(\text{End}(E_1 \oplus E_2))$ denote the curvature of ∇. Parts (i), (ii) below can be done in either order, either independently or with one part helping to do the other.

- (i) Notation and data as in part (b). Let $\hat{F}_\nabla$ and $\hat{F}^{E_i}$ $(i = 1, 2)$ denote the matrix-valued two-forms representing F_∇ and F^{E_i} with respect to the given local trivialization. Express $\hat{F}_\nabla$ in terms of $\hat{F}^{E_1}$ and $\hat{F}^{E_2}$.
- (ii) Express F_∇ in terms of F^{E_1} and F^{E_2} via a formula that does not involve connection-forms.

2. Let ∇ be a connection on a vector bundle $E \to M$. Let U, V be intersecting open sets in M over which E is trivial, let $(s_U), (s_V)$ be bases of sections of $E|_U, E|_V$ (with $(s_U) = \{s_{U,\mu}\}_{\mu=1}^k$ etc.), and let $g_{UV} \in C^\infty(U \bigcap V, GL(k, \mathbf{R}))$ be the corresponding transition function (with $(s_V) = (s_U)g_{UV}$). Let $\hat{F}_U, \hat{F}_V$ be the corresponding matrix-valued 2-forms representing the curvature F_∇ over U, V. Using each of the approaches indicated below (independently), show that on $U \bigcap V$, we have

$$\hat{F}_V = g_{UV}^{-1} \, \hat{F}_U \, g_{UV} \ . \tag{1}$$

Approach (i): F_∇ is a two-form on M with values in $\text{End}(E)$.
Approach (ii): In terms of connection forms Θ_U, Θ_V, we have

$$\hat{F}_U = d\Theta_U + \Theta_U \wedge \Theta_U \tag{2}$$

(etc. for V), and the connection forms are related on $U \bigcap V$ by

$$\Theta_V = g_{UV}^{-1} \, \Theta_U \, g_{UV} + g_{UV}^{-1} \, dg_{UV} \ . \tag{3}$$

(Equations (2) and (3) were derived in class. For the case $E = TM$, you derived (1) in Assignment 2,Problem 2, last semester, but I didn't ask you to hand it in, so I'm repeating the [generalized] problem here.) Thus, even though derivatives of the transition function appear at the level of connection-forms in (3), they disappear at the level of curvature-forms in (2)—as they must, in view of Approach (i).

3. Let $E \to M$ be a vector bundle, and let $\mathcal{A}(E)$ denote the set of all connections on E. As shown in class, $\mathcal{A}(E)$ naturally has the structure of an affine space whose group of translations is the vector space $\Omega^1(\mathrm{End}(E))$: given any connection ∇ on E,

$$\mathcal{A}(E) = \left\{\nabla + \eta \mid \eta \in \Omega^1(\mathrm{End}(E))\right\}. \tag{4}$$

As a reminder: in (4), the operator $\nabla + \eta : \Gamma(E) \to \Gamma(E \otimes T^*M)$ is defined by

$$(\nabla + \eta)_X(s) = \nabla_X s + \langle \eta, X\rangle(s) \quad \forall s \in \Gamma(E), X \in \Gamma(TM).$$

Show that for any $\nabla \in \mathcal{A}, \eta \in \Omega^1(\mathrm{End}(E))$, the curvatures of ∇ and $\nabla + \eta$ are related by

$$F_{\nabla+\eta} = F_\nabla + d_\nabla \eta + \eta \wedge \eta. \tag{5}$$

(Note that each of the three terms on the right-hand side of (5) is an $\mathrm{End}(E)$-valued two-form. The term $d_\nabla \eta$ is the covariant exterior derivative of the $\mathrm{End}(E)$-valued 1-form η, where the connection on $\mathrm{End}(E)$ is the one induced by the connection on E. The term $\eta \wedge \eta$ can also be written as $\frac{1}{2}[\eta, \eta]$, where $[\cdot, \cdot]$ is the "wedge-bracket" operation on $\mathrm{End}(E)$-valued differential forms.)

4. Let $E \to M$ be a vector bundle $F \subset E$ a sub-bundle. Note that, in general, a connection ∇ on E does not restrict to a connection on F: given $s \in \Gamma(F)$, $X \in \Gamma(TM)$, the covariant derivative $\nabla_X s$ is automatically a section of E, but its value at a given $p \in M$ need not lie in F_p.

(a) Using "∇", as usual, also to denote the connection induced by ∇ on any tensor-power of E, show that this induced connection ∇ *does* preserve the sub-bundles $\mathrm{Sym}^m(E)$ and $\bigwedge^m(E)$, $m \geq 2$.

(b) Let E be given a Riemannian metric h (in the vector-bundle sense: for $p \in M$, h_p is an inner product on E_p.). Let $\pi_F : E \to F$ be the bundle-homomorphism whose restriction to each fiber E_p is the orthogonal-projection map $E_p \to F_p$. For $s \in \Gamma(E), X \in \Gamma(TM)$, define

$$\nabla^F_X s = \pi_F(\nabla_X s).$$

Show that ∇^F is a connection on F.

Remark: Part (b) is phrased in terms of *orthogonal* projection only for the sake of familiarity. The only role of the metric h is to define a projection operator $E_p \to F_p$ for each p. A more general version of this problem is: suppose that E_1, E_2 are complementary sub-bundles of E (so that $E = E_1 \oplus E_2$). Let $\pi_{E_1} : \Gamma(E) \to \Gamma(E_1)$ be the bundle-homomorphism whose restriction to each fiber E_p is the projection map $E_p \to E_1|_p$ determined by the splitting $E_p = E_1|_p \oplus E_2|_p$. For $s \in \Gamma(E_1), X \in \Gamma(TM)$, define $\nabla^{E_1}_X s = \pi_{E_1}(\nabla_X s)$. Then ∇^{E_1} is a connection on E_1. Note that if E *is* Riemannian, and F is a sub-bundle of E, we have $E = F \oplus F^\perp$, so the connection defined in part (b) above is a special case of this more general construction.

5. In this problem, we use the notation $\mathfrak{gl}(k, \mathbf{R})$ for the space $M_{k\times k}(\mathbf{R})$ of real $k \times k$ matrices, and the notation $\mathfrak{so}(k) \subset \mathfrak{gl}(k, \mathbf{R})$ for the subspace consisting of all antisymmetric matrices. The notation comes from the fact that $\mathfrak{gl}(k, \mathbf{R})$ and $\mathfrak{so}(k)$ are the Lie algebras of the Lie groups $GL(k, \mathbf{R})$ and its subgroup $SO(k)$, respectively.

For Riemannian vector bundle of rank k over a manifold M, and let ∇ be a connection on E. Just as in the case $E = TM$, we say that ∇ *respects* (or *preserves*) h if for all $X \in \Gamma(TM)$, $s_1, s_2 \in \Gamma(E)$, we have

$$X\left(h(s_1, s_2)\right) = h\left(\nabla_X s_1, s_2\right) + h\left(s_1, \nabla_X s_2\right).$$

Assume that ∇ respects h. Let $U \subset M$ be an open set over which E is trivial.

(a) Show that there exist a basis of sections $\{e_\mu\}_{\mu=1}^k$ of $E|_U$ that is orthonormal at each point (i.e. $\{e_\mu(p)\}_{\mu=1}^k$ is an orthonormal basis of E_p for all $p \in U$). With the same abuse of terminology as in "basis of sections", call such a set $\{e_\mu\}$ an *orthonormal basis of sections* of $E|_U$.

(b) Choose an orthonormal basis of sections of $E|_U$ and let $\Theta \in \Omega^1(U, \mathfrak{gl}(k, \mathbf{R})$ be the corresponding connection form. Let $\hat{F}_U \in \Omega^2(U, \mathfrak{gl}(k, \mathbf{R})$ be the corresponding matrix-valued 2-form representing $F \in \Omega^2(\mathrm{End}(E))$. Show that, in fact Θ and $\hat{F}_U$ take their values in the subspace $\mathfrak{so}(k)$; i.e., $\Theta \in \Omega^1(U, \mathfrak{so}(k))$ and $\hat{F}_U \in \Omega^2(U, \mathfrak{so}(k))$.

6. Let $E \to M$ be a vector bundle with connection $\nabla = \nabla^E$ and let ∇^M be a connection on M. Write ∇ for the induced connection on $E \otimes T^*M$. For $s \in \Gamma(E)$, the *covariant Hessian* of s is the object Hs defined by

$$Hs = \nabla\nabla s \quad \in \Gamma(E \otimes T^*M \otimes T^*M). \tag{6}$$

(Note that the inner "∇" in (6) is simply the connection ∇^E, but the outer "∇" is the tensor-product connection on $E \otimes T^*M$, depending both the connection ∇^E and the connection ∇^M.)

(a) Let $s \in \Gamma(E)$, $X, Y \in \Gamma(TM)$. Then $Hs(X, Y) := (Hs)(X, Y)$ is a section of E. Show that

$$Hs(X, Y) = \nabla_X \nabla_Y s - \nabla_{\nabla^M_X Y} s. \tag{7}$$

(b) Assume that the connection ∇^M is torsion-free. Show that the curvature F_∇ of ∇ on E satisfies

$$F_\nabla(X, Y)s = Hs(X, Y) - Hs(Y, X).$$

(Thus, in general, $Hs(X, Y)$ is not symmmetric in X and Y.)