Differential Geometry IV—MAT 6932/4930 —Spring 2016
Assignment 3

1. (Optional). Let M be a manifold, E a distribution on M. Let $E^\perp \subset T^*M$ be the subbundle defined by $E_p^\perp = \{\theta \in T_p^*M \mid \theta(X) = 0 \ \forall X \in E_p\}$ (the annihilator of E_p). Let $\mathcal{I} \subset \Omega^*(M)$ be the ideal generated by $\Gamma(E^\perp)$ (i.e. $\mathcal{I}$ is the space of linear combinations of differential forms of the form $\omega \wedge \theta$, where $\omega \in \Omega^*(M)$ and $\theta \in \Gamma(E^\perp)$.) Let $d\mathcal{I}$ be the image of $d : \mathcal{I} \to \Omega^*(M)$. Prove that E is involutive if and only if $d\mathcal{I} \subset \mathcal{I}$.

2. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. An inner product k on $\mathfrak{g}$ is called *Ad-invariant* if for all $h \in G$, $v, w \in \mathfrak{g}$ we have $k(\mathrm{Ad}_g(v), \mathrm{Ad}_g(w)) = k(v, w)$. (Equivalently, k is Ad-invariant if the image of the homomorphism $G \to \mathrm{End}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$, lies in the orthogonal group of the inner-product space $(\mathfrak{g}, k)$.)

(a) Let $\tilde{k}$ be a Riemannian metric on G. Consider the following three conditions:

(i) $\tilde{k}$ is left-invariant.

(ii) $\tilde{k}$ is right-invariant.

(iii) $k := \tilde{k}_e$ is Ad-invariant.

Show that if any two of these conditions are satisfied, then so is the third.

Remarks: (1) A tensor field on G is called *bi-invariant* if it is both left-invariant and right-invariant. The most important cases are Riemannian metrics and differential forms. (2) The conclusion of (a) remains true if “Riemannian metric” is replaced by “tensor field”. You should be able to see this easily from your argument for part (a).

(b) Let $P \xrightarrow{\pi} M$ be a principal G-bundle. Suppose that $\mathfrak{g}$ has an Ad-invariant inner product k. Show that k naturally determines an inner product on each fiber of the associated vector bundle Ad, P, smoothly varying with the basepoint. (I.e. k determines a Riemannian structure, in the vector-bundle sense, on $\mathrm{Ad}P$.)

3. Let G be a Lie group, $P \xrightarrow{\pi} M$ be a principal G-bundle, and ρ a representation of G on a finite-dimensional vector space V. Recall that the *contragredient* or *dual* representation ρ^d of G on V^* is defined by setting $\langle \rho^d(g)\xi, v\rangle = \langle \xi, \rho(g^{-1})v\rangle$ for all $g \in G, \xi \in V^*, v \in V$.

(a) For $g \in G$, the map $V^* \times V^* \to V^* \otimes V^*$ given by $(\xi, \eta) \mapsto (\rho^d(g)\xi) \otimes (\rho^d(g)\eta)$ is bilinear, and hence defines a linear map $V^* \otimes V^* \to V^* \otimes V^*$. Let $(\rho^d \otimes \rho^d)(g)$ denote this map. Show that (i) $\rho^d \otimes \rho^d$ is a representation of G on $V^* \otimes V^*$, and (ii) for all $g \in G, k \in V^* \otimes V^*$, and $v, w \in V$, we have

$$\left((\rho^d \otimes \rho^d)(g)k\right)(v, w) = k(\rho(g^{-1})v, \rho(g^{-1})w). \quad (1)$$

(b) Show that for all $g \in G$, $(\rho^d \otimes \rho^d)(g)$ preserves the subspace $\mathrm{Sym}^2(V^*) \subset V^* \otimes V^*$. Hence $\rho^d \otimes \rho^d$ restricts to a representation of G on $\mathrm{Sym}^2(V^*)$, the space of symmetric bilinear forms on V. Show that for all $g \in G$, $(\rho^d \otimes \rho^d)(g)$ additionally preserves $\mathrm{Sym}^2_+(V^*)$ the subset of positive-definite elements of $\mathrm{Sym}^2(V^*)$ (the set of inner products on V).

We will refer to the above representation $\rho^d \otimes \rho^d$ to $\mathrm{Sym}^2(V^*)$ as *the representation of* G *on* $\mathrm{Sym}^2(V^*)$ *induced by* ρ.

(c) Let $E = E_\rho = P \times_\rho V$ be the vector bundle associated to P by the representation ρ. Exhibit canonical isomorphisms (i) $P \times_{\rho^d} V^* \to E^*$ and (ii) $P \times_{\rho^d \otimes \rho^d} (\mathrm{Sym}^2(V^*)) \to \mathrm{Sym}^2(E^*)$. Show that the latter isomorphism carries the associated fiber sub-bundle $P \times_{\rho^d \otimes \rho^d} (\mathrm{Sym}^2_+(V^*)) \subset P \times_{\rho^d \otimes \rho^d} (\mathrm{Sym}^2(V^*))$ to $\mathrm{Sym}^2_+(E^*) \subset \mathrm{Sym}^2(E^*)$, the fiber sub-bundle of $\mathrm{Sym}^2(E^*)$ whose fiber over $x \in M$ is $\mathrm{Sym}^2_+(E^*_x)$.

Remark: Hence there is a canonical 1–1 correspondence between the set of Riemannian metrics (in the vector-bundle sense) on E and the set of sections of $P \times_{\rho^d \otimes \rho^d} (\mathrm{Sym}^2_+(V^*))$.

4. Riemannian submersions. Let M^m, N^n be manifolds, and $\pi : N \to M$ a surjective submersion. Although (N, π, M) need not be a fiber bundle, for $x \in M$ we will still call $\pi^{-1}(x)$ the *fiber over* x. Recall that, by the Regular Value Theorem, every fiber of π is a submanifold of N of dimension $n - m$. At every $p \in M$, we define the *vertical subspace* $\mathcal{V}_p \subset T_pN$ to be $\ker(\pi_{*p})$, which is easily seen also to be the tangent space at p to the fiber containing p. A vector field V on N is called vertical if $V_p \in \mathcal{V}_p$ for all $p \in N$.

Assume now that we are given, additionally, a Riemannian metric $\tilde{g}$ on N. At each $p \in N$, we define the *horizontal space* $\mathcal{H}_p \subset T_pN$ to be the $\tilde{g}$-orthogonal complement of the vertical space $\mathcal{V}_p$. In particular, we have $T_pN = \mathcal{V}_p \oplus \mathcal{H}_p$, and $\pi_{*p}|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism (just as we have for horizontal spaces defined in the context of principal bundles).

Assume, finally, that we are also given a Riemannian metric g on M. We call π a *Riemannian submersion* $(N, \tilde{g}) \to (M, g)$ if, for all $p \in N$, the isomorphism $\pi_{*p}|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)}M$ is an isometry from the inner-product space $(\mathcal{H}_p, \tilde{g}_p)$ to the inner-product space $(T_{\pi(p)}M, g_{\pi(p)})$.[1]

Henceforth assume that π is a Riemannian submersion $(N, \tilde{g}) \to (M, g)$.

Some terminology and notation to be used below. For any vector field X on M, we define the *horizontal lift* of X to be the unique vector field $\tilde{X}$ on N for which $\tilde{X}_p \in \mathcal{H}_p$ and $\pi_{*p}\tilde{X}_p = X_{\pi(p)}$ for all p in N. Thus $\tilde{X}$ and X are π-related.

[1] Note that this does not say that $\tilde{g} = \pi^* g$; the positive-semidefinite tensor field $\pi^* g$ has a (generally) nontrivial null-space at each $p \in N$, namely the vertical space $\mathcal{V}_p$. (However, if π is a covering map, then $\mathcal{V}_p = \{0\}$, and we do have $\tilde{g} = \pi^* g$.)

More generally, call a (not necessarily horizontal) vector field $\overline{X}$ on N a *lift* of a vector field X on M if $\overline{X}$ and X are π-related. Recall that if vector fields $\overline{X}, \overline{Y}$ on N are π-related to vector fields X, Y on M, then

$$\pi_*[\overline{X}, \overline{Y}] = [X, Y]. \tag{2}$$

Note that any vertical vector field on N is a lift of the zero vector field on M. By (2), if V is vertical and $\overline{X}$ is a lift of X, then $\pi_*[V, \overline{X}] = [0, X] = 0$, so $[V, \overline{X}]$ is vertical.

Below, for any vector field X on M, we write $\tilde{X}$ for its horizontal lift. We will generally use letters X, Y, Z, W for vector fields on M, but use V for vertical vector fields on N.

For each $p \in N$, let $\mathrm{vert}_p : T_pN \to \mathcal{V}_p$ denote orthogonal projection to the vertical space (equivalently, the projection to $\mathcal{V}_p$ determined by the splitting $T_pN = \mathcal{V}_p \oplus \mathcal{H}_p$). Let vert denote the induced map $\Gamma(TN) \to \Gamma(\mathcal{V}) := \{\text{vertical vector fields on } N\}$. Similarly, define the horizontal projections hor_p and hor.

Let $\nabla, \tilde{\nabla}$ denote the Levi-Civita connections of the metrics $g, \tilde{g}$ respectively.

(a) Let X, Y be vector fields on M. Show that $\mathrm{hor}([\tilde{X}, \tilde{Y}]) = \widetilde{[X, Y]}$, and hence that

$$[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} + \mathrm{vert}([\tilde{X}, \tilde{Y}]). \tag{3}$$

(b) Let X, Y be vector fields on M and let V be a vertical vector field on N. Show that $V\left(\tilde{g}(\tilde{X}, \tilde{Y})\right) = 0$.

(c) Let X, Y be vector fields on M. Using the "six-term formula" for Levi-Civita connections, show that

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}\mathrm{vert}([\tilde{X}, \tilde{Y}]). \tag{4}$$

(d) Let X, Y be vector fields on M and let V be a vertical vector field on N. Show that

$$\tilde{g}(\tilde{\nabla}_V \tilde{X}, \tilde{Y}) = -\frac{1}{2}\tilde{g}(\mathrm{vert}([\tilde{X}, \tilde{Y}]), V). \tag{5}$$

(e) Let $\tilde{R}$ and R denote the Riemann tensors of $\tilde{g}, g$ respectively. Let $x \in M$, $p \in \pi^{-1}(x)$. Let X, Y, Z, W be vector fields on M. Show that

$$\begin{aligned} \tilde{g}_p(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) &= g_x(R(X, Y)Z, W) + \frac{1}{4}\tilde{g}_p(\mathrm{vert}([\tilde{X}, \tilde{Z}]), \mathrm{vert}([\tilde{Y}, \tilde{W}])) \\ &\quad - \frac{1}{4}\tilde{g}_p(\mathrm{vert}([\tilde{Y}, \tilde{Z}]), \mathrm{vert}([\tilde{X}, \tilde{W}])) + \frac{1}{2}\tilde{g}_p(\mathrm{vert}([\tilde{Z}, \tilde{W}]), \mathrm{vert}([\tilde{X}, \tilde{Y}])). \end{aligned} \tag{6}$$

(f) For vector fields X, Y on M, let $K(X,Y) = g(R(X,Y)Y, X)$. (Thus if $\{X_x, Y_x\}$ is an orthonormal set in T_xM, then $K(X,Y)|_x = \sigma(\mathcal{P}_{X_x,Y_x})$, the sectional-curvature function of (M,g) evaluated on the two-plane $\mathcal{P}_{X_x,Y_x}$ spanned by $\{X_x, Y_x\}$.) Analogously define $K(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X})$. Deduce from (6) that for $x \in M$, $p \in \pi^{-1}(x)$,

$$K(X,Y)|_x = K(\tilde{X},\tilde{Y})|_p + \frac{3}{4}\|\mathrm{vert}_p([\tilde{X},\tilde{Y}])\|^2. \tag{7}$$

Thus, for any two-plane $\mathcal{P} \subset T_xM$, letting $\widetilde{\mathcal{P}} \subset \mathcal{H}_p$ denote the horizontal lift of $\mathcal{P}$ (i.e. the plane $(\pi_{\mathcal{H}_p})^{-1}(\mathcal{P})$, we have $\sigma(\mathcal{P}) \geq \sigma(\widetilde{\mathcal{P}})$. Stated loosely (and without necessary hypotheses): sectional curvature goes up when we take a quotient.

5. Let $(P,\tilde{g})$, (M,g) be Riemannian manifolds. Call a map $\pi : P \to M$ a *principal Riemannian submersion* if (i) for some Lie group G, the space P is a principal G-bundle over M with projection π, (ii) π is a Riemannian submersion $(P,\tilde{g}) \to (M,g)$, and (iii) for each $h \in G$, the map $R_h : (P,\tilde{g}) \to (P,\tilde{g})$ is an isometry. Since $(R_h)_*$ maps vertical spaces to vertical spaces, (iii) implies that $(R_h)_*$ maps also maps horizontal spaces (as defined for Riemannian submersions) to horizontal spaces, and does so isometrically. Then the distribution $\mathcal{H}$ defined by the horizontal spaces of the Riemannian submersion is invariant under G, hence is a connection A on P. Call this the "natural connection" for the principal Riemannian submersion.

Let $\pi : (P,\tilde{g}) \to (M,g)$ be a principal Riemannian submersion with group G, let A be the natural connection on P, and let $\tilde{F}_A \in \Omega^2(P;\mathfrak{g})$ be the curvature 2-form of A.

(a) Deduce from problem 4 that for all $x \in M, p \in \pi^{-1}(x)$, we have

$$K(X,Y)|_x = K(\tilde{X},\tilde{Y})|_p + \frac{3}{4}\|\iota_p(\tilde{F}_A(X,Y))\|^2_{\tilde{g}}. \tag{8}$$

(Here $\iota_p : \mathfrak{g} \to \mathcal{V}_p$ has the same meaning as in class.) Thus the sectional curvature of (M,g) is, loosely speaking, the sectional curvature of $(P,\tilde{g})$ increased by the squared norm of the curvature of the natural connection of the principal bundle $P \to M$.

Note that the G-invariance of the metric $\tilde{g}$ implies that for $p, q \in \pi^{-1}(x)$ we have $K(\tilde{X},\tilde{Y})|_p = K(\tilde{X},\tilde{Y})|_q$. Thus the first term on the right-hand side of (8) depends only on x, and hence so also does the second term.

(b) For each $p \in P$, define an inner product $\tilde{k}_p$ on $\mathfrak{g}$ by $\tilde{k}_p = \iota_p^*(\tilde{g}_p)$. Let Ad' denote the representation of G on $\mathrm{Sym}^2(\mathfrak{g}^*)$ induced by the adjoint representation Ad of G on $\mathfrak{g}$. Show that for all $p \in P, h \in G$, we have

$$\tilde{k}_{p\cdot h} = \mathrm{Ad}'(h^{-1})\tilde{k}_p\ . \tag{9}$$

Hence $\tilde{k}$ is canonically identified with a section k of $P \times_{\mathrm{Ad}'} (\mathrm{Sym}^2(\mathfrak{g}^*))$, positive-definite at every point. From the Remark at the end of Problem 3, k is canonically identified with a Riemannian metric (in the vector-bundle sense) on $\mathrm{Ad}\,P$. (This generalizes Problem 2b.)

(c) Let $F_A \in \Omega^2(M; \mathrm{Ad}P)$ be the curvature 2-form of A, viewed as a bundle-value form on M. Let k be the Riemannian structure on $\mathrm{Ad}\,P$ defined in part (b). Show that (8) simplifies even further than in part (a), to

$$K(X,Y)|_x = K(\tilde{X},\tilde{Y})|_p + \frac{3}{4}\| F_A(X,Y)|_x \|_k^2 \ , \tag{10}$$

where $x = \pi(p)$. Thus the sectional curvature of (M, g) is, loosely speaking, the "horizontal" sectional curvature of $(P, \tilde{g})$ increased by the (squared norm of the) curvature of the natural connection of the principal bundle $P \to M$ (where the norm is natural as well).

6. (Optional). Let G be a Lie group, $P \xrightarrow{\pi} M$ be a principal G-bundle, ρ a representation of G on a finite-dimensional vector space V, $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ the induced Lie-algebra homomorphism. Let $E = P \times_\rho V$ be vector bundle associated to P by ρ, let A be a connection on P, and let ∇ be the induced connection on E. Let $\omega_A \in \Omega^1(P; \mathfrak{g})$ the connection form of A.

Let $U \subset M$ be open and let $\mathbf{e} = \{e_1, \dots, e_k\}$ be a basis of sections of $E|_U$. Call $\mathbf{e}$ *ρ-admissible* if there exists a basis $\mathbf{v} = \{v_1, \dots, v_k\}$ of V and a section s of $P|_U$ such that for all $x \in U$, $e_i(x) = [(s(x), v_i)], 1 \le i \le k$, where [ ] is the equivalence relation on $P \times V$ defining the space $P \times_\rho V$. (Obviously, $P|_U$ must be trivial for s to exist, hence for $\mathbf{e}$ to be ρ-admissible.)

Let $U \subset M$ be open; assume that $\mathbf{e}$ is a ρ-admissible basis of sections of $E|_U$, and let $s, \mathbf{v}$ be as in the definition of "ρ-admissible" above. Let $\Theta^{(\mathbf{e})} \in \Omega^1(U; M_{k\times k}(\mathbf{R}))$ be the connection form of ∇ with respect to the local basis of sections $\mathbf{e}$. Let $L^{\mathbf{v}} : \mathrm{End}(V) \to M_{k\times k}(\mathbf{R})$ be the map carrying an endomorphism of V to its matrix with respect to the basis $\mathbf{v}$.

(a) Show that

$$\Theta^{(\mathbf{e})} = L^{\mathbf{v}} \circ \dot{\rho} \circ s^*\omega_A \ . \tag{11}$$

(b) Let $\tilde{F}_A \in \Omega^2(P; \mathfrak{g})$ be the curvature 2-form of A, let $F_\nabla \in \Omega^2(M; \mathrm{End}(E))$ be the curvature of ∇, and let $F_\nabla^{(\mathbf{e})} \in \Omega^2(U; M_{k\times k}(\mathbf{R}))$ be the corresponding matrix-valued 2-form obtained by using the basis $\mathbf{e}(x)$ to identify an endomorphism of E_x with a matrix ($x \in U$). Show that

$$F_\nabla^{(\mathbf{e})} = L^{\mathbf{v}} \circ \dot{\rho} \circ s^*\tilde{F}_A \ . \tag{12}$$

(c) (This can be done with or without using parts (a) and (b).) Show that $\dot{\rho}$ determines a vector-bundle homomorphism $\dot{\rho}_E : \mathrm{Ad}\,P \to \mathrm{End}(E)$. Letting $F_A \in \Omega^2(M; \mathrm{Ad}\,P)$ denote the curvature 2-form of A, viewed as a bundle-valued form on M, show that

$$F_\nabla = \dot{\rho}_E \circ F_A \ . \tag{13}$$