Convex Sets in Vector Spaces

Throughout these notes, "vector space" means "real vector space".

Although vector spaces are defined purely algebraically, we often visualize them geometrically. Geometry in Euclid's sense—in which *lengths* and *angles* play key roles—requires not just a vector-space structure, but an *inner product*, about which you have learned or will learn in this course. However, certain more primitive geometric notions, such as *point*, *line*, and another that is the chief topic of these notes—*convex set*—make sense in any vector space, not just in inner-product spaces.

Definition 1 (Lines). Let V be a vector space and let $\mathbf{v}, \mathbf{u} \in V$ with $\mathbf{u} \neq \mathbf{0}$. The *line through* $\mathbf{v}$ *in the direction of* $\mathbf{u}$ (or *parallel to* $\mathbf{u}$) is the set

$$\ell = \{\mathbf{v} + t\mathbf{u} \mid t \in \mathbf{R}\}. \tag{1}$$

A subset of V is a *line* (or *straight line*) if it is a line through some element of V in some direction.

Exercise

1. To make sure you have a feel for where the terminology comes from, draw examples of lines in $\mathbf{R}^2$ and $\mathbf{R}^3$ through various points in various directions.

The next several exercises develop some simple but important features of lines.

Exercises Below, V is a fixed but arbitrary vector space.

2. Show that if ℓ is a line in V then for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \ell$, the vectors $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_4$ are parallel (i.e. one is a multiple of the other).

3. Show that every one-dimensional subspace of V is a line, but that if $\dim(V) \geq 2$, the converse is false.

4. Show that for any line ℓ in V, there are infinitely many vectors $\mathbf{v}$ and nonzero vectors $\mathbf{u}$ such that ℓ is the line through $\mathbf{v}$ in the direction of $\mathbf{u}$.

Definition 2 (Translation of a set by a vector). Let $\mathbf{v} \in V$ and let S be a nonempty subset of V. The *translate of* S *by* $\mathbf{v}$ (or S *translated by* $\mathbf{v}$) is the set

$$\mathbf{v} + S := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in S\}.$$

If $S' = \mathbf{v} + S$ for some $\mathbf{v} \in V$, we say that S' is a *translate of* S.

Exercise

5. (a) Show that "is a translate of" is an equivalence relation. I.e. show that if S, S', S'' are nonempty subsets of V, then (i) S is a translate of S; (ii) if S' is a translate of S, then S is a translate of S', and (iii) if S' is a translate of S, and S'' is a translate of S', then S'' is a translate of S.

   (b) Let S be a nonempty subset of V and let $\mathbf{v}_1, \mathbf{v}_2 \in V$.

   (i) Show that
$$\mathbf{v}_2 + (\mathbf{v}_1 + S) = (\mathbf{v}_2 + \mathbf{v}_1) + S. \tag{2}$$

   In other words: translation by $\mathbf{v}_1$, followed by translation by $\mathbf{v}_2$, has the same effect as transation by $\mathbf{v}_2 + \mathbf{v}_1$.

   Because of equation (2), we can unambiguously use the notation "$\mathbf{v}_2 + \mathbf{v}_1 + H$" for either side of the equation; the two conceivable interpretations are equal.

   (ii) Show that if $S' = \mathbf{v}_1 + S$, then $S = (-\mathbf{v}_1) + S'$.

   (c) Show that for every line ℓ in V, there is a unique one-dimensional subspace $L \subseteq V$ such that ℓ is a translate of L. Further, show that in the context of Exercise 4, the vectors $\mathbf{u}$ are exactly the nonzero elements of L.

6. Let H be a subspace of a vector space.

   (a) Let $\mathbf{v} \in V$. Show that $\mathbf{v} \in \mathbf{v} + H$.

   (b) (i) Let $\mathbf{v} \in H$. Show that $\mathbf{v} + H = H$.

   (ii) Conversely, show that if $\mathbf{v} \in V$ and $\mathbf{v} + H = H$, then $\mathbf{v} \in H$.

   (Thus, (i) and (ii) together show that if $\mathbf{v} \in V$, then $\mathbf{v} + H = H$ iff $\mathbf{v} \in H$.)

7. Let V be a vector space $\mathbf{v}_1, \mathbf{v}_2 \in V$, let H_1, H_2 be subspaces of V, and suppose that $\mathbf{v}_1 + H_1 = \mathbf{v}_2 + H_2$. Show that $H_1 = H_2$. (*Hint*: problems 5b and 6a.)

Definition 3 (Translated subspaces). A *translated subspace*[1] in a vector space V is any set of the form $\mathbf{v} + H$ where $\mathbf{v} \in V$ and H is a subspace of V. Exercise 7 shows that if A is a translated subspace in V, then there is a *unique* subspace $H = H_A$ of which A is a translate. (However, unless A consists of a single element, there are infinitely many vectors by which H can be translated to obtain A.) The subspace H_A is sometimes called the *tangent space*, the *underlying subspace*, or the *director space* of A. We define $\dim(A)$, the *dimension* of the translated subspace A, to be the dimension of its tangent space.

[1] Other common terminology for this is *affine subspace*. However, this terminology can be misleading to the linear-algebra student since an affine subspace is generally *not* a subspace in the sense of linear algebra—an affine subspace need not be closed under addition or scalar multiplication, and need not contain the zero vector.

Exercise 5c above shows that "translated subspace" generalizes the notion of "line"; lines are simply 1-dimensional translated subspaces.

Note that subspaces are a special case of translated subspaces: a subspace H is the translate of H by the zero vector (or by any other element of H).

Because we often visualize vector spaces geometrically, we also often refer to elements of vector spaces as *points*[2], and say that "the line ℓ passes through $\mathbf{v}$" if $\mathbf{v} \in \ell$. This terminology will be used frequently in the remainder of these notes. Exercise 8a below is the origin of the term "*line*ar transformation".

<u>Exercises</u>

8. (a) Show that under any linear transformation, the image of a line is either a line or a single point. (Said more lengthily: Let V, W be vector spaces, $\ell \subset V$ a line, and $T : V \to W$ a linear transformation. Then $T(\ell) := \{T(\mathbf{v}) \mid \mathbf{v} \in \ell\}$ is either a line in W or a single point of W.)

   (b) More generally, show that under any linear transformation, the image of a translated subspace A is a translated subspace whose dimension is at most $\dim(A)$.

9. Let V be a vector space.

   (a) Show that for any two distinct elements $\mathbf{v}_1, \mathbf{v}_2 \in V$, there is a unique line ℓ passing through $\mathbf{v}_1$ and $\mathbf{v}_2$, and that all of the following are true:

$$
\begin{aligned}
\ell &= \text{the line through } \mathbf{v}_1 \text{ in the direction } \mathbf{v}_2 - \mathbf{v}_1 \\
&= \{(1-t)\mathbf{v}_1 + t\mathbf{v}_2 \mid t \in \mathbf{R}\} \qquad (3) \\
&= \{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid t_1, t_2 \in \mathbf{R},\ t_1 + t_2 = 1\}. \qquad (4)
\end{aligned}
$$

   (b) How is the line passing through $\mathbf{v}_1$ and $\mathbf{v}_2$ related to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$? Are they the same sets? Is one a subset of the other?

   (c) Draw an example in $\mathbf{R}^2$ showing a line through two points, characterized as in (3), and indicating the portions of the line corresponding to (i) $t = 0$, (ii) $t = 1$, (iii) $t < 0$, (iv) $t > 1$, (v) $t = 1/2$, (vi) $t = 1/3$ and $t = 2/3$, and (vii) $0 < t < 1$.

Definition 4 (Line segments). Let V be a vector space, $\mathbf{v}_1, \mathbf{v}_2 \in V$. The *line segment between* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ is the set

$$
\begin{aligned}
&\{(1-t)\mathbf{v}_1 + t\mathbf{v}_2 : 0 \le t \le 1\} \qquad (5) \\
= &\{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 : t_1, t_2 \in \mathbf{R},\ t_1 + t_2 = 1,\ t_1 \ge 0,\ t_2 \ge 0\}. \qquad (6)
\end{aligned}
$$

The points $\mathbf{v}_1$ and $\mathbf{v}_2$ are called the *endpoints* of this line segment.

[2]The student is warned that although in these notes, and in other contexts, elements of vector spaces may be casually referred to as points, the terms "vector" and "point" are in general not interchangeable. There are many instances in which it is important to make a distinction between points and vectors; e.g. in Calculus 3 a point in space is conceptually different from a three-component vector.

Note that in equation (6), the conditions imposed on the t_i imply that $t_i \in [0,1]$ for each i. Thus the conditions on the t_i in (6) could equivalently be written as "$t_1 + t_2 = 1,\ 0 \le t_1 \le 1,\ 0 \le t_2 \le 1$." Conventionally, the "$\le 1$"s are omitted from the definition just because they're superfluous. Nonetheless, it's helpful to keep in mind that the restrictions in equation (6) restrict each t_i to lie in $[0,1]$.

Exercise

10. Let $a, b \in \mathbf{R}$ with $a < b$. Prove that the line segment between a and b is exactly the closed interval $[a,b]$.

    Warning: You will be tempted to ask "Isn't this obvious?" and may be unsure what this exercise is requiring you to do. The problem is that there are two different ways you've seen the term "line segment" used: the way you've used it all your life (Euclid's geometric notion of a line segment) and the way it's used in Definition 4 (a purely algebraic definition, unknown to Euclid). Calling the object in Definition 4 a line segment is no guarantee that this algebraic definition reproduces the geometric notion we're used to, any more than calling the object in Definition 4 a chicken would have helped it lay eggs. So you are being asked here to prove that $[a,b]$ satisfies Definition 4. It's not hard, but you have to *do* it.

    This sort of potential confusion arises frequently for students of mathematics. Something is given a name—often a word that already has some standard meaning or connotation—in hindsight, *after* some mathematician has proved the object has certain properties that the standard usage implies. But there's no such thing as "proof by terminology" or "proof by notation". Any time the usage of a familiar word is extended to apply to a context new to the student, the student has to go through what the name-giving mathematician went through the first time, proving that the object in question has properties that justify the way it's now named.

Note that in Definition 4, unlike in Exercise 9, we do not require that the points $\mathbf{v}_1, \mathbf{v}_2$ be distinct. If $\mathbf{v}_2 = \mathbf{v}_1$, the line segment is *degenerate*: it consists of a single point. Of course, if $\mathbf{v}_2 \neq \mathbf{v}_1$, then comparing (3) and (5), or (4) and (6), shows that the line segment between $\mathbf{v}_1$ and $\mathbf{v}_2$ is a subset of the line passing through $\mathbf{v}_1$ and $\mathbf{v}_2$, and Exercise 10 and your work on Exercises 9c should convince you that the words "between" and "endpoints" in Definition 2 are appropriate.

Exercise

11. Show that linear transformations map line segments to line segments. In such a context, how are the endpoints of the image line-segment related to the endpoints of the domain line-segment? Show that it is possible for the image line-segment to be degenerate even if the domain line-segment is non-degenerate.

Definition 5 (Convex set). A subset A of a vector space V is *convex* if for all $\mathbf{v}_1, \mathbf{v}_2 \in A$, the line segment between $\mathbf{v}_1$ and $\mathbf{v}_2$ is contained in A.

__**Examples**__ (stated without proof; you will provide some of the proofs in exercises below)

1. Any interval in $\mathbf{R}$ (whether open, closed, or half-open/half-closed) is convex.
2. Every line, and line segment, in $\mathbf{R}^3$ is convex.
3. The interior, or interior-plus-boundary, of any triangle or parallelogram in $\mathbf{R}^2$ is convex.
4. Any circular or elliptical disk (the interior, or interior-plus-boundary, of a circle or ellipse in $\mathbf{R}^2$) in $\mathbf{R}^2$ is convex. Similarly the interior, or interior-plus-boundary, of a sphere or ellipsoid in $\mathbf{R}^3$ is convex. All of these are examples in which convexity is "geometrically obvious" but surprisingly tricky to show algebraically. One of the shortest and nicest proofs involves something called a *convex function*, which you may learn about in an Advanced Calculus class or certain graduate classes, but which we will not discuss in these notes since it is not a purely linear-algebraic object. It is also easier to prove that the sets in this example are convex after you've been introduced to inner-product spaces.
5. The interior, or interior-plus-boundary, of a dumbbell-shaped region in $\mathbf{R}^3$ is not convex.
6. The unit circle in $\mathbf{R}^2$, $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 : x^2 + y^2 = 1 \right\}$, is not convex. Neither is any other simple closed curve.
7. Trivially, any vector space V is a convex subset of itself.

__**Exercises**__

12. A *closed parallelepiped* P in $\mathbf{R}^n$ is any set of the form

$$\{\mathbf{v}_0 + \sum_{i=1}^{n} t_i \mathbf{a}_i : 0 \le t_i \le 1,\ 1 \le i \le n\},$$

where $\mathbf{v}_0, \mathbf{a}_i \in \mathbf{R}^n,\ i = 1 \ldots n$. If the set $\{\mathbf{a}_i\}_{i=1}^n$ is linearly independent (hence a basis of $\mathbf{R}^n$) then P is *nondegenerate* ("solid"); if $\{\mathbf{a}_i\}_{i=1}^n$ is linearly dependent then P is degenerate (it has dimension $< n$, in the sense of Definition 10 later in these notes). Show that any closed parallelepiped is convex.

Of course, one can define "open parallelepiped" using strict inequalities, and show that these are convex as well. Note that for $n = 1$, a nondegenerate parallelepiped is a closed interval; for $n = 2$ it's a parallelogram; and for $n = 3$ it's the object you learned to call "parallelepiped" in Calculus 3.

13. Show that every translate of a convex set in a vector space is convex.

14. (a) Show that every subspace of a vector space is convex.

    (b) Show that every translated subspace of a vector space is convex. (Of course, (a) is a special case of (b).)

15. Show that linear transformations map convex sets to convex sets. (Said more lengthily: Let V, W be vector spaces, $A \subseteq V$ convex, and $T : V \to W$ a linear transformation. Then $T(A) := \{T(\mathbf{v}) : \mathbf{v} \in A\}$ is convex.)

16. Shown that in any vector space, the intersection of two convex sets is convex, but that the union of convex sets need not be convex.

17. Generalize the first part of Exercise 16: prove that in any vector space the intersection of arbitrarily many convex sets (whether finitely many, infinitely but countably many, or uncountably many) is convex. (Mathematicians would often write the statement this way: if $\mathcal{A}$ is an arbitrary nonempty collection of convex sets, then $\bigcap_{A\in\mathcal{A}} A$ is convex.)

Definition 6 (Convex linear combinations). A *convex linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ $(n \geq 1)$ in a vector space is any vector of the form

$$\sum_{i=1}^{n} t_i \mathbf{v}_i \,, \quad \text{where } \sum_{i=1}^{n} t_i = 1 \text{ and } \ t_i \geq 0 \ \text{ for each } i.$$

(Thus the line segment between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ consists of all convex linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$; see equation (4).)

Exercise

18. Show that if A is a convex subset of a vector space, then any convex linear combination of elements of A lies in A.

Convex Hulls, Convex Polyhedra, and Simplices

Definition 7 (Convex hull). Let S be a nonempty subset of a vector space V. The *convex hull* of S in V is the intersection of all convex sets that contain S. (Said another way: the convex hull of S in V is $\bigcap_{A\in\mathcal{A}} A$, where $\mathcal{A}$ is the collection of all convex subsets of V that contain S. Note that this collection is always nonempty, since V itself is a convex subset of V that contains S.)

By Exercise 17, the convex hull of any nonempty subset of a vector space is convex.

Exercise

19. Show that if V is a subspace of a vector space W, and S is a nonempty subset of V, then the convex hull of S in V coincides with the convex hull of S in W. (*Hint*: Exercise 14.)

In view of the result of Exercise 19, for nonempty $S \subseteq V$, we usually just use the terminology "convex hull of S", omitting the "in V", since the convex hull does not change if we choose to regard S as a subset of some larger vector space containing V. Below, we incorporate this fact into our notation, writing Hull(S) for the convex hull of S, without any reference to the vector space in which S is assumed to lie.

Exercises

20. Show that a convex set is its own convex hull. (I.e. if A is a convex subset of a vector space, then $\mathrm{Hull}(A) = A$.)

21. Show that if $S \subseteq S' \subseteq \{\text{some vector space}\}$, then $\mathrm{Hull}(S) \subseteq \mathrm{Hull}(S')$.

22. Let S be a nonempty subset of a vector space, let $\mathbf{v} \in \mathrm{Hull}(S)$, and let $S' = S \bigcup \{\mathbf{v}\}$. Show that $\mathrm{Hull}(S') = \mathrm{Hull}(S)$. (If you are unable to prove this for *arbitrary* nonempty sets S, try to prove it at least for all *finite* nonempty sets S.)

23. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a finite nonempty subset of a vector space V.
    (a) Show that Hull(S) is precisely the set of convex linear combinations of the $\mathbf{v}_i$.
    (b) For $V = \mathbf{R}^2$, draw at least one example illustrating the convex hull of 2, 3, 4, and 5 points in "general position" (one example for each number of points). Here, "general position" means that no three points are collinear. For the case of four points, draw two examples: one in which one of the points lies in the convex hull of the set consisting of the other three points, and another in which none of the points lies in the convex hull of the set consisting of the other three points.

24. Generalize Exercise 23a: show that for *any* (not necessarily finite) nonempty subset S of a vector space V, the convex hull of S is precisely the set of convex linear combinations of elements of S, i.e.

$$\{\mathbf{w} \in V : \exists\, n \geq 1 \text{ and } \mathbf{v}_1, \ldots, \mathbf{v}_n \in S \text{ such that } \mathbf{w} \text{ is a convex linear combination of the } \mathbf{v}_i\}.$$

25. Let S be as in Exercise 23, and for $1 \leq i \leq n$, define $\mathbf{w}_i = \mathbf{v}_i - \mathbf{v}_1$. Let $H = \mathrm{span}(\{\mathbf{w}_i\}_{i=1}^n)$, a subspace of V. Show that the subspace H does not depend on the labeling of the vectors in S; i.e. if, when defining the $\mathbf{w}_i$, "$\mathbf{v}_1$" is replaced by any of the other vectors in S (assuming $n \geq 2$, so that there *are* other vectors), the span of the new $\{\mathbf{w}_i\}$ is the same as the span of the old.

Definition 8 (Convex polyhedron). A *convex polyhedron*, *convex polytope*[3], or *finitely-generated convex set* is the convex hull of a finite nonempty set of vectors. If a convex

[3]Some authors prefer to reserve the word "polygon" for the two-dimensional case, and "polyhedron" for the three-dimensional case, but there is no general agreement (for the precise meaning of "dimension" here, see Definition 10). There are also other definitions of convex polyhedra/polytopes in the literature, but most, if not all, are equivalent to Definition 8.

polyhedron P is the convex hull of $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, we say that P is *generated* (as a convex set) by the $\mathbf{v}_i$; S is called a *generating set* (or *set of generators*) for P, and the $\mathbf{v}_i$ are called *generators*.

Note that this usage of "generate" for convex sets is *not* the same as what "generate" means for subspaces. A convex subset of $\mathbf{R}^2$ could have 100 generators, even though no more than two of them could be linearly independent.

A natural question is, "Can a convex set have more than one set of generators?" The answer is "yes", and Exercise 22 above shows us a way to produce examples. E.g. if P is a triangle in $\mathbf{R}^2$, then the vertices of P generate P. But if we throw in any other point, or several other points, of P (whether in the interior or on one of the edges), we get a larger generating set. Intuitively, in this example the vertices of P are a "minimal" generating set in some sense: we can't get away with fewer generators, and the vertices form the only three-element generating set. The next definition gives us a preliminary definition of "minimal" that, as the exercises below will show, has consequences as strong as what we intuitively see in the triangle example.

Definition 9. Let P be a convex polyhedron (see Definition 8) generated by a finite set S in some vector space. We call S a *minimal* generating set if no element can be deleted from S without changing the convex hull; i.e. if for every nonempty proper subset S' of S we have $\text{Hull}(S') \neq \text{Hull}(S) = P$.

Exercise 21 above shows that if we delete any elements from a set of generators S, obtaining a smaller set of generators S', then $\text{Hull}(S) \supset \text{Hull}(S')$. Therefore deleting an element or elements from a generating set leads either to a smaller convex hull or to the same convex hull as before; this operation never *adds* to the convex hull. In particular, if S has at least two elements, and for every nonempty subset S' obtained from S by deleting a single element we have $\text{Hull}(S') \neq \text{Hull}(S)$, then for any nonempty subset obtained by deleting *more* than one element we still have $\text{Hull}(S') \neq \text{Hull}(S)$. Thus in Definition 9, we would not have altered the meaning of "minimal generating set" had we restricted attention to subsets S' differing from S by a single element. This is really the way to *think* of what Definition 9 is saying, but wording the definition this way would have made it lengthier. Mathematicians prefer to keep their definitions as short as possible—or at least as short as possible without confusing someone a first-time reader. An alternative definition that is not lengthier, but makes it less clear exactly what sense of "minimal" is intended, is "Let P be the convex hull of a finite set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in some vector space. We call S a *minimal* generating set if for $1 \leq i \leq n$, $\mathbf{v}_i \notin \text{Hull}(S'_i)$, where S'_i is the set obtained by deleting $\mathbf{v}_i$ from S." (To show that this is equivalent to Definition 9, use the result of Exercise 22.)

Note that Definition 9 leaves open the possibility that a given finitely-generated convex set P could have two minimal sets of generators with different numbers of elements. We will see in Exercise 26 below that not only can there never be two minimal sets of generators with different numbers of elements, there never be two different minimal sets of generators *at all*. Therefore we don't have to add these extra conditions to our definition of "minimal"; they are implied by the simple definition above. Thus, a generating set for

P in the sense of Definition 9, is also minimal in the sense of cardinality.

Exercises

26. (*Warning*: this exercise is *much* longer and harder than any other exercise in these notes.) Let V be a vector space, $P \subseteq V$ a convex set for which each of the finite sets $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, $S' = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, is a minimal generating set. Show (a) that $m = n$, and (b) that $S = S'$ (i.e. the $\mathbf{w}_i$ are the same as the $\mathbf{v}_i$, just possibly listed in a different order).

    Of course, part (b) implies part (a), but (a) is easier than (b); you may find that you are able to prove (a) without successfully proving the full result (b).

To recap, a finite nonempty set S of vectors uniquely determines a convex polyhedron. Exercise 26 in some sense asserts the converse: that a convex polyhedron *uniquely* determines a minimal set of generators, so we can unambiguously refer to *the* minimal set of generators. These generators have the name you would expect:

Definition 10. The elements of a minimal generating set of a convex polyhedron P are called its *vertices* (singular: *vertex*).

Definition 11. Let P be a convex polyhedron in a vector space V and let H_P be the subspace of V called "H" in Exercise 25. We define $\dim(P)$, the *dimension* of P, to be the dimension of the subspace H_P.

Exercises

27. Why is the result of Exercise 25 crucial to Definition 11?

28. Let P be a convex polyhedron generated by $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Show that each of the generators $\mathbf{v}_i$ lies in P. In particular, P contains all of its vertices.

29. Let P be a convex polyhedron in a vector space V. Show that there is a unique translated subspace A of minimal dimension containing P, and that $\dim(P) = \dim(A)$.

    (The "minimal dimension" condition is needed for the following reason. A translated subspace A of dimension less than $\dim(V)$ is always contained in at least one larger translated subspace, namely V itself. If $\dim(A) < \dim(V) - 1$, then there are infinitely many translated subspaces that strictly contain A. Thus a convex polyhedron P in V will be contained in more than one translated subspace if $\dim(P) < \dim(V)$, and in infinitely many if $\dim(P) < \dim(V) - 1$. However, the list of *dimensions* of translated subspaces will have a smallest element, and the problem above asserts that among all the translated subspaces containing P, only one of them has this smallest dimension.)

30. Let P be a convex polyhedron with n vertices. Show that $\dim(P) \leq n - 1$.

31. Show that a nondegenerate closed parallelepiped P in $\mathbf{R}^n$ is a convex polyhedron. What is dimension of P? How many vertices does P have, and what are they?

32. Let V, W be vector spaces, $T : V \to W$ a linear transformation, and $P \subseteq V$ a convex polyhedron.

    (a) Show that T maps P to a convex polyhedron $T(P)$ generated by the images of the vertices of P.

    (b) Show that if T is invertible, then the vertices of $T(P)$ are precisely the images under T of the vertices of P.

    (c) Give an example showing the relevance of invertibility in part (b). You should be able to produce a simple example with $V = \mathbf{R}^2$ and $W = \mathbf{R}$.

Definition 12. For $n \geq 0$, an *n-simplex* is a convex polyhedron having $n+1$ vertices and dimension n (the maximum possible dimension for this number of vertices, by Exercise 30). A *simplex* (plural: *simplices*) is any set that is an n-simplex for some n.

Simplices are tremendously useful in many areas of pure and applied mathematics, from topology to optimization.

Examples

1. A 0-simplex is a point; a 1-simplex is a line segment; a 2-simplex is a triangle (see Exercise 33 below); a 3-simplex is a tetrahedron.

2. The *standard n-simplex* is the set

$$\sigma_n = \left\{ \begin{pmatrix} t_1 \\ \vdots \\ t_{n+1} \end{pmatrix} \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} t_i = 1, \text{ all } t_i \geq 0 \right\}.$$

   Its vertices are the standard unit basis vectors of $\mathbf{R}^{n+1}$ (one vertex lies on each coordinate axis).

Exercises

33. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be non-collinear points in $\mathbf{R}^2$. Convince yourself that the triangle Δ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a 2-simplex with these vertices. (Hint: First show that Δ contains each of the line segments whose endpoints are two of the $\mathbf{v}_i$. Then consider the line segment joining $\mathbf{v}_3$ to a variable point on the line segment between $\mathbf{v}_1$ and $\mathbf{v}_2$.) In this exercise, you are asked only to "convince yourself", rather than to "show" or "prove", because you have not been given an algebraic definition of a triangle (in particular, of the region interior to the boundary).

34. If a vector space V contains an n-simplex, what is the smallest possible dimension of V?

35. Let $n \geq 1$ and let P be a convex polyhedron with vertices $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Show that the following are equivalent:

   (a) For *some* i_0 $(1 \leq i_0 \leq n)$, the $(n-1)$-element set $\{\mathbf{v}_i - \mathbf{v}_{i_0} : 1 \leq i \leq n,\ i \neq i_0\}$ is linearly independent.

   (b) For *every* i_0 $(1 \leq i_0 \leq n)$, the $(n-1)$-element set $\{\mathbf{v}_i - \mathbf{v}_{i_0} : 1 \leq i \leq n,\ i \neq i_0\}$ is linearly independent.

   (c) P is a simplex.

36. For $n > 1$, is a nondegenerate closed parallelepiped in $\mathbf{R}^n$ ever a simplex?