Inductive proofs: some common mistakes and misconceptions

Let's start with an example of a correct, well written, inductive proof. I've highlighted in blue a sentence I'll refer to later. Below, $\mathbf{N}$ denotes the set of *natural numbers* (positive integers).

Example 1. Prove that for all $n \geq 1$, the sum of the first n natural numbers is $n(n+1)/2$.

Proof: For each $n \in \mathbf{N}$, let $P(n)$ be the statement that the sum of the first n natural numbers is $n(n+1)/2$. $P(1)$ asserts that 1=1(1+1)/2, which is true.

Suppose now that m is any natural number for which $P(m)$ is true. Then the sum of the first $m+1$ natural numbers is $\frac{m(m+1)}{2} + (m+1) = (m+1)(\frac{m}{2}+1) = (m+1)\frac{m+2}{2} = (m+1)(m+2)/2$, so $P(m+1)$ is true. By induction, it follows that $P(n)$ is true for all $n \in \mathbf{N}$. ■

If you're wondering why I introduced a second variable-name, m, instead of sticking with n throughout, see the "Side note" at the end of this handout.

There is more than one correct way to write this proof (or other inductive proofs). But there are also numerous incorrect ways, some of which I'll discuss below. But first, let's review some terminology.

"Base case", inductive *step*, and inductive *hypothesis*

Suppose we are given an assertion "Statement $P(n)$ is true for all (integers) $n \geq n_0$," a typical candidate for proof-by-induction.[1] An inductive proof of this assertion has two basic steps: (i) proving, or checking, that $P(n_0)$ is true (the "base case") and (ii) proving that whenever $n \geq n_0$ is an integer for which $P(n)$ is true, $P(n+1)$ is also true (the *inductive step*). Once these steps are completed, we wrap up the proof with a sentence such as "By induction, statement $P(n)$ is true for all $n \geq n_0$."

The inductive *step* starts with the inductive *hypothesis*, e.g. "Let $n \geq n_0$ be such that statement $P(n)$ is true." (In the proof in Example 1, the blue sentence is the inductive hypothesis.) This hypothesis is then used to show that $P(n+1)$ is true, completing the inductive *step*. (In the proof in Example 1, the sentence after the blue one completes the inductive step.) **The inductive *step* and the inductive *hypothesis* are not the**

[1] Of course, in any specific example, my wording-form "Statement $P(n)$ is true" is replaced by wording that's appropriate for the specific fact being proved, and that need not contain the words "statement" or "is true". Generally, context makes clear that the only values of n we're considering are integers, in which case we allow ourselves to say "for all $n \geq n_0$" rather than "for all integers $n \geq n_0$."

same thing. You should not refer to the inductive hypothesis as the inductive step, or vice-versa.

As you may have noticed, in the sample proof in Example 1, the terms "base case" and "inductive step" did not appear. A common misconception among students is that these terms are *supposed to* appear in an inductive proof. No; explicitly *labeling* these steps does not affect the validity of an inductive proof. (However, if your proof is badly written, then labeling the steps may make the proof more decipherable. So until you're reasonably good at writing inductive proofs, you may be better off including these step-labels. Including them will not turn a poorly written proof into a well-written one, but at least may give the reader extra clues as to what you were trying to say.)

The terms *base case* and *inductive step* are, essentially, "training wheels" that are used to help *teach* the structure of a proof-by-induction[2]. They are virtually never used in higher mathematics, or by mathematicians communicating with each other. However, the term "inductive *hypothesis*" is used by mathematicians. For example, if one or more sentences comes between the inductive hypothesis and *applying* this hypothesis, wording like "By the inductive hypothesis, [such-and-such is true]" is commonly used.

It is possible that your Sets and Logic instructor *required* you to put the "base case" and "inductive step" labels into your inductive proofs (to make sure you understood the steps, and could structure an inductive proof correctly). But once you've gotten the hang of these proofs, you can drop these labels.

The inductive hypothesis: good and bad wording

Assume that the assertion we're trying to prove by induction is is, "Statement $P(n)$ is true for all $n \geq n_0$." One *good, clear* way to write the inductive hypothesis is

$$\text{"Let } n \geq n_0 \text{ be such that statement } P(n) \text{ is true."} \tag{1}$$

I'll give some other valid options later, but first I want to go through some common *mistakes*. Improper wording can turn an intended inductive hypothesis into no hypothesis at all, or into too strong a hypothesis.

1. Suppose for simplicity that $n_0 = 1$ and that you've already established the "base case". An all-too-common (and proof-invalidating) way of beginning the inductive step is, "Assume that $P(n)$ is true for some $n \in \mathbf{N}$" (or, equivalently, "Assume that for some $n \in \mathbf{N}$, the statement $P(n)$ is true").[3] Since $P(1)$ is already known to be true, **the statement "$P(n)$ is true for *some* $n \in \mathbf{N}$" is already *known* to be**

[2]To my knowledge, they are also historically recent inventions; I never saw them before the 1990s.

[3]Unfortunately, even some math instructors and textbook authors this mistake—generally without realizing or accepting that it is a *mistake*—and may even *teach* students to write inductive hypotheses this *objectively wrong* way. The widely used Abstract Algebra textbook from which I am currently teaching (spring 2024) makes this mistake.

true; it is not something that needs to be *assumed*. Nor could assuming that $P(n)$ is true for *some* $n \in \mathbf{N}$ ever help you prove that $P(n)$ is true for *all* $n \in \mathbf{N}$. The inductive step consists of proving that whenever $P(n)$ is true (i.e. for *every* $n \in \mathbf{N}$ for which $P(n)$ is true), $P(n+1)$ is true as well. The "…for some $n \in \mathbf{N}$" hypothesis does not yield the needed "whenever-ness".

Someone who habitually makes this type of mistake might now respond to this criticism with, "But when I said, 'Assume that $P(n)$ is true for some $n \in \mathbf{N}$,' of course I didn't mean just for one particular n, like $n = 1$; I meant that n could be any natural number for which $P(n)$ is true." **Tough. That is not what "Assume that $P(n)$ is true for some $n \in \mathbf{N}$" *means* in the English language.** Unambiguous wording exists, and should have been used, for the *precise* idea that you meant to express. It is always *your* responsibility, when writing, to say *exactly* what you mean, and to say it clearly and unambiguously. It is *never* the reader's responsibility (even when the reader is your teacher!) to read your mind and say to him/herself "Oh, the writer can't possibly mean what he/she said; that would just be stupid. So I'll take the writer to have meant what he/she *should* have said, rather than what he/she *did* say."

2. A variant of the mistake above is to word the inductive hypothesis as "Assume that $P(n)$ is true for any $n \in \mathbf{N}$." When appearing *after* a statement being quantified, the quantifiers "for any", "for all," "for every", and "for each", *all mean the same thing*. (This is usually true when the quantifiers appear *before* the statement being quantified, as well, but there are exceptions involving "for any", one of which I give an example of after this list of mistakes.) The above "for any" hypothesis means exactly the same thing as "Assume that $P(n)$ is true for every $n \in \mathbf{N}$," so the assumption in this variant amounts to *assuming exactly what you're supposed to be proving*. This, of course, completely invalidates your proof.

3. A couple of other mistakes in inductive hypotheses involve using the words "for" and "where" as quantifiers. **"For" and "where" are not quantifiers.**

   (a) Consider the wording, "Assume that $P(n)$ is true for $n \in \mathbf{N}$." This wording of the inductive hypothesis has the same problem that "Assume that $P(n)$ is true for any $n \in \mathbf{N}$" had above: it amounts to *assuming exactly what you're supposed to be proving*, thereby invalidating your proof. Although "for" is not a quantifier, what "for $n \in \mathbf{N}$" means (whether or not you want it to) is "for *all* $n \in \mathbf{N}$." I.e. when "for" is used as a *substitute* for a quantifier (or an *abbreviated version of* for a quantifier), the meaning is always "for *all*", never "for *some*".

   Note: The use of "for" as an abbreviated version of "for all" is actually fairly common.

   (b) Consider the wording, "Assume that $P(n)$ is true, where $n \in \mathbf{N}$." This is even worse than the "… for $n \in \mathbf{N}$" example. In the above use of "where", the

> reader can't even *guess* what the writer means; there is no default meaning of "where" as an abbreviation of, or substitute for, a quantifier.

The way I wrote the inductive hypothesis earlier (sentence (1))—i.e. starting with an appropriate "Let" statement—is one of a few *correct* ways to write an inductive hypothesis. I find this way to be the *cleanest, clearest* way, though it is a bit stilted. A couple of other valid ways of writing an inductive hypothesis—but which come perilously close to being invalid (especially the second one), and which **which I do not recommend for students**—are

$$\text{“Suppose, for some } n \geq n_0\text{, that } P(n) \text{ is true”} \tag{2}$$

and

$$\text{“Suppose, for any } n \geq n_0\text{, that } P(n) \text{ is true.”} \tag{3}$$

Why are these okay? Didn't I label usages of "for some $n \geq n_0$" and "for any $n \geq n_0$" as *mistakes* above? Answer: **Word-order makes a difference!!** (And so do punctuation and general sentence-structure.) In sentences (2) and (3), because of the word-order and punctuation, the "for some $n \geq n_0$" and "for any $n \geq n_0$" are modifying "Suppose" rather than "$P(n)$ is true." In the examples I gave earlier, the word-order implied that "for some $n \geq n_0$" and "for any $n \geq n_0$" were modifying "$P(n)$ is true."

Again, I **do not recommend** that students use the last two examples as models; it is too easy to make some slight alteration that changes the meaning.

Side note: In the proof in Example 1, you may have wondered why I introduced the letter m instead of just sticking with n throughout. The reason is that I wrote the concluding "By induction ..." sentence in the same paragraph as the inductive step. It would have been a bit confusing to use the notation n in both the inductive step and the conclusion; the two n's would have had different meanings in consecutive sentences within the same paragraph. But if I'd put the concluding sentence into its own, new paragraph, it would have been okay to use the same letter in both paragraphs, because the inductive step was over after "so $P(m+1)$ is true;" this step's role for the notation m, or whatever letter I'd used, could have been regarded as dying at the end of that sentence. The main reason I did not write the sample proof that way (with the concluding sentence in a new paragraph) is that too many students aren't used to sectioning their proofs into paragraphs, and would not have understood that the paragraph-break makes a difference. But the paragraphing problem is not the topic of this handout.