Difference Between Inverse Functions and Inverse Images

Not every function has an inverse function. In order for $f : X \to Y$ to have an inverse, f must be one-to-one and onto. However, for ANY function, the inverse image of ANY subset of the codomain is defined. Unfortunately, the notation for inverse function is part of the notation for inverse image, so that you have to determine from context which meaning is meant. The basic definition of inverse image is

Definition: Let $f : X \to Y$ be a function. Let $U \subseteq Y$. Then the *inverse image of* U *under* f, denoted $f^{-1}(U)$, is the set

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}.$$

Note that it is only the entire character-string "$f^{-1}(U)$" that is defined here; the sub-string "f^{-1}" has no independent meaning in this context (just as the letter d in *dog* has no independent meaning). Note also that the inverse image of a set is a SET, not an element (although it is possible for the inverse image of a set to contain only one element).

It's important to realize that you do not need a function to be 1-1 or onto for inverse images to make sense.

Examples. Take $X = Y = \mathbf{R}$, and let $f(x) = \cos x$. Then

1. $f^{-1}(1)$ makes no sense, since "1" is not a subset of the codomain. (It is an *element* of the codomain.)
2. $f^{-1}(\{1\}) = \{$all integer multiples of $2\pi\}$.
3. $f^{-1}([0,1]) = \bigcup_{n\in\mathbf{Z}}[2n\pi - \pi/2, 2n\pi + \pi/2]$. (Recall that $\mathbf{Z}$ is the standard notation for the set of all integers, and that $[a,b]$ means the closed interval in $\mathbf{R}$ from a to b.)
4. $f^{-1}(\{2\}) = \emptyset$ (LaTeX's symbol for the empty set, not the number zero).
5. $f^{-1}((0,63]) = \bigcup_{n\in\mathbf{Z}}(2n\pi - \pi/2, 2n\pi + \pi/2)$.
6. $f^{-1}([-1,1]) = \mathbf{R}$.

If a function $f : X \to Y$ happens to have an inverse function f^{-1}, then the notation "$f^{-1}(U)$" may seem ambiguous: does it mean the *inverse* image of U under the function f, or the *image* of U under the function f^{-1} ? (The *image* of a set under a function is defined in the "Sets and Functions" handout.) Fortunately, both interpretations yield the same set: treating the notation $f^{-1}(U)$ as meaning the inverse image of U

under f, (i) if $y \in U$, then $f(f^{-1}(y)) = y \in U$, so $f^{-1}(y) \in f^{-1}(U)$, and consequently $\{f^{-1}(y) \mid y \in U\} \subseteq f^{-1}(U)$; while (ii) for any $x \in f^{-1}(U)$, the element $f(x)$ lies in U, so $x = f^{-1}(f(x)) \in \{f^{-1}(y) \mid y \in U\}$. (Hence $f^{-1}(U) = \{f^{-1}(y) \mid y \in U\}$; i.e.

$$\text{(inverse image of } U \text{ under } f) = \text{(image of } U \text{ under the function } f^{-1})$$

if f happens to have an inverse function). In particular, applying this when U consists of a single element of Y: if f has an inverse, then

$$f^{-1}(\{y\}) = \{f^{-1}(y)\} \quad \text{for each } y \in Y.$$

Make sure you understand exactly why the curly braces are where they are (on each side of the equation).

Despite what I said in the example with "$f^{-1}(1)$" earlier, it is common to "abuse terminology" or "abuse notation" and refer to the inverse image of a single element, when one what actually means is the object illustrated in the second example. Even with this abuse of terminology, the inverse image of an element is understood to be a *set*. If $f : X \to Y$ is a function and $y \in Y$, then any element of the set $f^{-1}(y)$ (abuse of notation!—this should really be written "$f^{-1}(\{y\})$") is called *an* inverse image of y. For example, for the cosine function, 4π is an inverse image of 1.