Difference Between Inverse Functions and Inverse Images

Not every function has an inverse function. In order for $f : X \to Y$ to have an inverse, f must be one-to-one and onto. However, for ANY function, the inverse image of ANY subset of the codomain is defined. Unfortunately, the notation for inverse function is part of the notation for inverse image, so that you have to determine from context which meaning is meant. The basic definition of inverse image is

Definition: Let $f : X \to Y$ be a function. Let $B \subseteq Y$. Then the *inverse image* (or *pre-image*) *of* B *under* f, denoted $f^{-1}(B)$, is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Note that it is only the entire character-string "$f^{-1}(B)$" that is defined here; the sub-string "f^{-1}" has no independent meaning in this context (just as the letter d in *dog* has no independent meaning). Note also that the inverse image of a set is a SET, not an element (although it is possible for the inverse image of a set to contain only one element).

It's important to realize that you do not need a function to be 1-1 or onto for inverse images to make sense.

Examples. Take $X = Y = \mathbf{R}$, and let $f(x) = \cos x$. Then

1. $f^{-1}(1)$ makes no sense, since "1" is not a subset of the codomain. (It is an *element* of the codomain.)
2. $f^{-1}(\{1\}) = \{$all integer multiples of $2\pi\}$.
3. $f^{-1}([0,1]) = \bigcup_{n\in\mathbf{Z}}[2n\pi - \pi/2, 2n\pi + \pi/2]$. (Recall that $\mathbf{Z}$ is the standard notation for the set of all integers, and that $[a, b]$ means the closed interval in $\mathbf{R}$ from a to b.)
4. $f^{-1}(\{2\}) = \emptyset$. (This is LaTeX's notation for the empty set, not zero.)
5. $f^{-1}((0, 63]) = \bigcup_{n\in\mathbf{Z}}(2n\pi - \pi/2, 2n\pi + \pi/2)$.
6. $f^{-1}([-1, 1]) = \mathbf{R}$.

Despite what I said in the first example, mathematicians frequently "abuse terminology" or "abuse notation" and refer to the inverse image of a single element y, when what is actually meant is the object illustrated in the second example, the inverse image of the corresponding singleton-set $\{y\}$. I.e., given a function $f : X \to Y$ and an element $y \in Y$, when mathematicians are being careless they may write $f^{-1}(y)$ when they mean

$f^{-1}(\{y\})$.[1] Even when the notation $f^{-1}(y)$ is (ab)used as notation for $f^{-1}(\{y\})$, the phrase "inverse image of y" is understood to mean $f^{-1}(\{y\})$.[2]

Given a function $f : X \to Y$ and an element $y \in Y$, each element of $f^{-1}(\{y\})$ is called *an* inverse image (or a *pre-image*) of y. For example, for the cosine function on the previous page, 4π is an inverse image of 1. So is -62π.

How are inverse functions related to inverse images?

Remember that inverse *functions* don't always exist, but inverse *images* always *do* exist. ***If* a function $f : X \to Y$ happens to have an inverse function** (denoted f^{-1}, as usual, with the notation pronounced "f-inverse"), then the following statement—which may be confusing to read—is true: for any $B \subseteq Y$,

$$f^{-1}(B) = \{f^{-1}(y) \mid y \in Y\}. \tag{1}$$

In other words, **<u>if</u>** f has an inverse function, then for any subset $B \subseteq Y$, **the *inverse image* of** B **under** f equals **the *image* of** B **under** f^{-1}**.** (Recall that for a function $g : Z \to W$ and a subset $A \subseteq Z$, the *image of A under g* is the set $g(A) := \{g(x) : x \in A\}$.) The reason equation (1) may be confusing to read is that in it, the character-string "f^{-1}" is used twice, with a different meaning each time: the first use of "f^{-1}" is part of the notation for the inverse image of a set, while the second use refers to the function called "f-inverse". (*Exercise*: Show that if f has an inverse function, the equality (1) does, indeed, hold for every $B \subseteq Y$.)

<u>If</u> f has an inverse function, **and** B contains just a single element of Y, then the notion of *inverse image* reproduces the notion of *inverse function*:

$$f^{-1}(\{y\}) = \{f^{-1}(y)\} \text{ for each } y \in Y.$$

Make sure you understand exactly why the curly braces are where they are (on each side of the equation).

[1] This does not mean that it's okay for *students* to do this! The right to take that liberty needs to be earned by proving, unequivocally, over a long period of time, that you understand what you *should* really be writing.

[2] *Note mostly for instructors*: When communicating with students who are (relatively) new to the "inverse image" concept, instructors should not take the liberty of writing "$f^{-1}(y)$" when "$f^{-1}(\{y\})$" is meant, as this can be very confusing for the student. It is true that if the instructor at least *says*, or *writes, explicitly* the phrase "inverse image" when writing "$f^{-1}(y)$", then a student paying close attention may realize that the instructor actually meant "$f^{-1}(\{y\})$". But this is a distraction for the student, caused unnecessarily by the instructor, and may lead to the student not hearing what the instructor says next.