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Algebra in Lagrange Multiplier Problems

When faced with solving several simultaneous equations in several unknowns, one way to proceed is successively to eliminate variables. For example, if you are trying to extremize a function f of two variables x, y, z subject to a constraint $g(x, y) = c$, you wind up having to solve three equations in the three unknowns x, y, λ. Two of these equations come from $\nabla f = \lambda \nabla g$; the third is the constraint $g(x, y) = c$. Generally it makes sense to eliminate λ first, if possible, and then successively to eliminate the other variables until all solutions are found. The examples below illustrate this method. In the first example, there is only one case to consider.

With practice and experience you will have to write out only a small fraction of the steps I've written below, and will find other ways to short-cut some of the work. To demonstrate, I'll do the first example below methodically (taking almost a page), then efficiently (taking three lines).

<u>Example 1.</u> Minimize $f(x, y) = x+y$ for positive x, y subject to the constraint $\frac{1}{x} + \frac{1}{y} = 10$. Assume the minimum exists.

Here $g(x, y) = \frac{1}{x} + \frac{1}{y}$, so the **i** and **j** components of the vector equation $\nabla f = \lambda \nabla g$, together with the constraint equation, give us

$$\begin{aligned} 1 &= \lambda \frac{-1}{x^2}, \\ 1 &= \lambda \frac{-1}{y^2}, \\ \frac{1}{x} + \frac{1}{y} &= 10. \end{aligned}$$

Step 1a. Solve for λ in the first equation: $\lambda = -x^2$.
Step 1b. In the remaining two equations, everywhere you see a λ, replace it by $-x^2$:

$$\begin{aligned} 1 &= (-x^2)\frac{-1}{y^2} = \frac{x^2}{y^2}, \\ \frac{1}{x} + \frac{1}{y} &= 10. \end{aligned}$$

Step 1c. Simplify the above equations if possible

$$\begin{aligned} x^2 &= y^2, \\ \frac{1}{x} + \frac{1}{y} &= 10. \end{aligned}$$

Step 2a. Repeat the idea of Step 1: Choose a variable (let's say y) in the first of our new equations, and solve for it in terms of the other variable(s) (in this case x). Since we are given that x, y are positive, the only solution of $y^2 = x^2$ is $y = x$.

Step 2b. Substitute $y = x$ into the remaining equation we found in Step 1c.

$$\frac{1}{x} + \frac{1}{x} = 10.$$

Step 2c. Solve this equation for x: $\quad 2/x = 10$, so $x = \frac{2}{10} = \frac{1}{5}$.

Step 3. Now use the result of Step 2a to get y: $\quad y = x = \frac{1}{5}$.

Step 4. Write all the solution pairs (x, y) we've found (in this case there's just one), and compute f of them.

$$f(\frac{1}{5}, \frac{1}{5}) = \frac{2}{5}$$

Since we are assuming the minimum value exists, this must be it (2/5).

Comment. We did not bother to figure out the value of λ, since that's not needed for our final answer. That's a reason for eliminating λ first. ■

Example 1, redone less methodically but more efficiently. If you have trouble following what I'm about to do, or repeating it without looking at these notes, or doing harder problems, then you need more practice with the methodical approach.

Write down the original three equations in three unknowns. The first two equations imply $x^2 = y^2$, hence $x = y$. The third equation then gives $2/x = 10$, so $x = 1/5 = y$. Thus the minimum value of f is $1/5 + 1/5 = 2/5$.

Example 2. Minimize $f(x, y, z) = x^4 + 8y^4 + 27z^4$ for x, y, z subject to the constraint $x + y + z = 11/12$. Assume the minimum exists.

Step 0a. Write the relevant four equations in four unknowns (three from $\nabla f = \lambda \nabla g$, the other from the constraint).

$$\begin{aligned} 4x^3 &= \lambda, \\ 4 \cdot 8y^3 &= \lambda, \\ 4 \cdot 27z^3 &= \lambda, \\ x + y + z = \frac{11}{12}&. \end{aligned}$$

Step 0b (helpful but not necessary). Redefine λ, if possible, to simplify the algebra coming up. Specifically, it doesn't matter if you originally started with $\nabla f = \lambda \nabla g$ or with $\nabla f = c\lambda \nabla g$ where c is any constant; both equally well express the relation "∇f is an unknown multiple of ∇g. If we choose $c = 4$ (i.e. replace λ by 4λ in the equations above), we can divide through by the 4 and get the simpler equations

$$\begin{aligned} x^3 &= \lambda, \\ 8y^3 &= \lambda, \\ 27z^3 &= \lambda, \\ x+y+z &= \frac{11}{12}. \end{aligned}$$

Step 1. Solve the first equation for λ (getting $\lambda = x^3$), then substitute this into the remaining equations:

$$\begin{aligned} 8y^3 &= x^3, \\ 27z^3 &= x^3, \\ x+y+z &= \frac{11}{12}. \end{aligned}$$

Step 2a. Solve the (new) first equation for y in terms of x (in this case by taking the cube root of both sides): $2y = x$, hence $y = \frac{x}{2}$.

Step 2b. Substitute this expression for y wherever y occurs in the remaining two equations.

$$\begin{aligned} 27z^3 &= x^3, \\ x+\frac{x}{2}+z &= \frac{11}{12}. \end{aligned}$$

Step 3a. Solve the (new) first equation for z in terms of x (in this case by taking the cube root of both sides): $3z = x$, hence $z = \frac{x}{3}$.

Step 3b. Substitute this expression for z wherever z occurs in the remaining two equations.

$$x+\frac{x}{2}+\frac{x}{3} = \frac{11}{12}.$$

Step 4. Solve this equation for x:

$$x(1+\frac{1}{2}+\frac{1}{3}) = \frac{11}{6}x = \frac{11}{12} \quad \Rightarrow \quad x = \frac{1}{2}.$$

Step 5. Use the results of Steps 3a and 2a to get z and y from the x you just found:

$$z = x/3 = \frac{1}{6}, \quad y = x/2 = \frac{1}{4}.$$

Step 6. Plug all the solution triples (x, y, z) you just found (in this case there's just one) into f:

$$f(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}) = (\frac{1}{2})^4 + 8(\frac{1}{4})^4 + 27(\frac{1}{6})^4 = \frac{11}{96}.$$

<u>Example 2, redone less methodically but more efficiently.</u> Start with the four original equations. The first three give $x^3 = 8y^3 = 27z^3$, hence $x = 2y = 3z$, hence $y = x/2, z = x/3$. The fourth equation then gives $(11/6)x = 11/12$, hence $x = 1/2, y = 1/2, z = 1/6$. Hence the minimum value of f is $f(1/2, 1/4, 1/6) = 11/96$.

<u>Example 3</u>. Minimize $f(x, y, z) = x^2+2y^2+3z^2$ subject to the constraint $x^2+y^2+z^2 = 1$. Here I won't keep writing "Step 1, Step 2, ..."; you'll have to fill in the missing work on your own. The strategy is the same, but there is an additional complication: you can't immediately solve for λ; there are some special cases. After dividing the "$\nabla f = \lambda \nabla g$" equations by 2, we have

$$\begin{aligned} x &= \lambda x, \\ 2y &= \lambda y, \\ 3z &= \lambda z, \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

We can't immediately get λ from the first equation, because x might be zero. So there are two cases.

Case I. $x = 0$, or

Case II. $\lambda = 1$ (since if $x \neq 0$, we can divide through by x in the first equation.).

Analysis of Case I.

Substituting $x = 0$ into the lower three equations, we get

$$\begin{aligned} 2y &= \lambda y, \\ 3z &= \lambda z, \\ y^2 + z^2 &= 1. \end{aligned}$$

The first new equation now gives us two sub-cases:

Case IA. $y = 0$, or

Case IB. $\lambda = 2$.

Analysis of Case IA.

Subsituting $y = 0$, we obtain

$$\begin{aligned} 3z &= \lambda z, \\ z^2 &= 1. \end{aligned}$$

The first of these is now irrelevant to the question we're trying to answer, since the second equation immediately gives us $z = \pm 1$. So from Case IA we get the points $(0, 0, \pm 1)$ to plug into f (eventually).

Analysis of Case IB.

Plugging $\lambda = 2$ into the last two Case I equations, we have

$$\begin{aligned} 3z &= 2z, \\ y^2 + z^2 &= 1. \end{aligned}$$

The first equation gives $z = 0$, and plugging into the second we get $y = \pm 1$. So from Case IB we get the points $(0, \pm 1, 0)$ to plug into f (eventually). We've now exhausted Case I.

Analysis of Case II.

Substituting $\lambda = 1$ into the latter three of the original four equations, we obtain

$$\begin{aligned} 2y &= y, \\ 3z &= z, \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

The first two of the new equations give $y = 0$ and $z = 0$. Plugging into the third equation we get $x^2 = 1$, so $x = \pm 1$. This gives us the points $(\pm 1, 0, 0)$ to plug into f.

Our case-by-case analysis is now complete, so we compute f of all the points we've found:

$$\begin{aligned} f(1, 0, 0) &= 1. \\ f(-1, 0, 0) &= 1. \\ f(0, 1, 0) &= 2. \\ f(0, -1, 0) &= 2. \\ f(0, 0, 1) &= 3. \\ f(0, 0, -1) &= 3. \end{aligned}$$

Hence the minimum value of f is 1 and the maxiumum value is 3.

Example 3, redone less methodically but more efficiently. Starting with the original four equations in four unknowns, rewrite the first three as

$$\begin{aligned} x(\lambda - 1) &= 0, \\ y(\lambda - 2) &= 0, \\ z(\lambda - 3) &= 0. \end{aligned}$$

At a solution point (x, y, z) at most one of x, y, z can be nonzero, since if $x \neq 0$ then $\lambda = 1$; if $y \neq 0$ then $\lambda = 2$; and if $z \neq 0$ then $\lambda = 3$ (and λ cannot simultaneously equal two different numbers!). Therefore the solution points must have $x = y = 0$, or $x = z = 0$, or $y = z = 0$. From the constraint equation we get the value of the missing variable (± 1), so the set of solution points is $(0, 0, \pm 1)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. Plugging into f, the minimum value is 1 and the maximum value is 3.