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Partial Fractions and Laplace Transform Problems

The general method for using the Laplace transform to solve a linear differential equation $L[y] = g$ (with some initial conditions) is to (1) transform both sides of the equation, (2) solve for $Y(s)$, then (3) invert the transform to find $y(t)$. The most difficult part is generally step (3), which often involves rewriting $Y(s)$ using partial fractions in such a way that the result is a linear combination of terms that appear on the Laplace transform table. The examples presented in the textbook (Nagle & Saff) usually involve combining fractions between steps (1) and (2). While this approach is not wrong, often it results in more work than necessary, partly because a common denominator in $Y(s)$ can have very high degree, and partly because the whole purpose of partial fractions is to un-combine fractions and rip apart common denominators. Below is an example of how to do a problem without first combining fractions. To shorten this and other examples, it helps to know three frequently used simple partial fraction identities (which can easily be re-derived if you forget them):

1. $\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}(\frac{1}{x+a} - \frac{1}{x+b})$
2. $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}(\frac{1}{x-a} - \frac{1}{x+a})$ (this follows from previous line)
3. $\frac{x}{x^2-a^2} = \frac{x}{(x-a)(x+a)} = \frac{1}{2}(\frac{1}{x-a} + \frac{1}{x+a})$

Example. Use Laplace transforms to solve the initial-value problem

$$y'' - 4y = 4t - 8e^{-2t}, \; y(0) = 0, \; y'(0) = 5.$$

Method. Laplace-transforming this IVP gives $s^2Y - 5 - 4Y = 4/s^2 - 8/(s+2)$, so

$$(s^2-4)Y = 5 + \frac{4}{s^2} - \frac{8}{s+2},$$

and hence

$$Y = \frac{5}{s^2-4} + \frac{4}{s^2(s^2-4)} - \frac{8}{(s^2-4)(s+2)}. \tag{1}$$

One way to proceed is as in the textbook: combine fractions en route to equation (1), getting a 5th degree denominator, then set the resulting fraction equal to something of the form $A/s + B/s^2 + C/(s-2) + D/(s+2) + E/(s+2)^2$, then multiply out, then solve for A, B, C, D, E. The alternative I'm suggesting is to break expression (1) into three sub-expressions, and proceed as follows. Note that our basic identities easily handle two of these three sub-expressions, and for the third sub-expression we only have to deal with a cubic denominator instead of a 5th degree denominator.

$$\frac{5}{s^2-4} = 5\frac{1}{s^2-4} = \frac{5}{4}(\frac{1}{s-2} - \frac{1}{s+2}) \qquad \text{using identity 2.}$$

$$\begin{aligned}\frac{4}{s^2(s^2-4)} = 4\frac{1}{s^2(s^2-4)} &= \frac{1}{s^2-4} - \frac{1}{s^2} && \text{using identity 1 with } x = s^2 \\ &= \frac{1}{4}(\frac{1}{s-2} - \frac{1}{s+2}) - \frac{1}{s^2} && \text{using identity 2.}\end{aligned}$$

$$\frac{8}{(s^2-4)(s+2)} = \frac{8}{((s-2)(s+2))(s+2)} = \frac{8}{(s-2)(s+2)^2} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{(s+2)^2}.$$

Multiplying out,

$$8 = A(s+2)^2 + B(s-2)(s+2) + C(s-2).$$

Now use your favorite method to find A, B, C. (My favorite is to expand out the right-hand side and collect like powers of s, getting $8 = A(s^2+4s+4)+B(s^2-4)+C(s-2) = s^2(A+B)+s(4A+C)+(4A-4B-2C)$, and then equate coefficients of equal powers of s on the two sides of this equation, getting the simultaneous equations $A+B=0,\ 4A+C=0, 4A-4B-2C=8$, which I then solve.) When done correctly, you wind up with $A = 1/2, B = -1/2, C = -2$, so

$$\frac{8}{(s^2-4)(s+2)} = \frac{1/2}{s-2} - \frac{1/2}{s+2} - \frac{2}{(s+2)^2}.$$

Inserting these partial-fractions decompositions for our three sub-expressions into equation (1) and combining only terms with identical denominators,

$$Y = \frac{1}{s-2} - \frac{1}{s+2} + 2\frac{1}{(s+2)^2} - \frac{1}{s^2},$$

and hence, using the Laplace transform table,

$$y(t) = e^{2t} - e^{-2t} + 2te^{-2t} - t.$$