Exponentials of Matrices.

Suppose p is a polynomial of degree m: $p(x) = b_0 + b_1 x + \ldots + b_m x^m$ for some (real or complex) numbers $b_0, b_1, \ldots, b_m$. For any $n \times n$ square matrix A, we define the $n \times n$ matrix $p(A)$ by "plugging in A for x":

$$p(A) = b_0 I + b_1 A + \ldots + b_m A^m$$

where I is the $n \times n$ identity matrix. To shorten the notation we define $A^0 = I$ and write $p(A) = \sum_{k=0}^{m} b_k A^k$.

Example If $p(x) = 1 + x^2$, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. then $J^2 = -I$, so $p(J) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

A Taylor series can be thought of as an "infinite polynomial" $f(x) = \sum_{k=0}^{\infty} b_k x^k$. For any such series, and any $n \times n$ matrix A, we can define

$$f(A) = \sum_{k=0}^{\infty} b_k A^k$$

provided the series converges (i.e. if for each i and j, the series for the $(ij)^{th}$ entry of the series for $f(A)$ converges). Note that in general $f(A)$ is *not* the matrix whose $(ij)^{th}$ entry is $f(A_{ij})$, so you must be careful with notation: $(f(A))_{ij} \neq f(A_{ij})$.

Recall the following very important Taylor series, all of which converge for all real x:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \end{aligned}$$

It can be shown that the corresponding series for e^A, $\cos A$, $\sin A$ converge for all square matrices A.

Matrix exponentials are particularly important in the study of systems of linear differential equations. (See your textbook, pp. 262-264, for examples and for a slightly different approach to this subject.) It can be shown that for any square matrix A, e^{tA} is a differentiable matrix-valued function of t, with $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$. Using this, it is not hard to show that for a system $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$, where A is an $n \times n$ matrix and $\mathbf{y}$ is an $\mathbf{R}^n$-valued function of t with initial value $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbf{R}^n$, the solution of the corresponding initial-value problem is $\mathbf{y}(t) = e^{tA}\mathbf{y}(0)$. This generalizes the result for $n = 1$ that you learned in your differential equation class (or in the context of "exponential growth/decay" problems in Calculus 2).

<u>Exercise 1.</u> Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (a) Compute J^2, J^3, J^4, J^5. From the pattern you see, deduce what J^k is for all $k \geq 0$. (b) Compute the matrix e^{tJ}, where t is a general real number (do *not* choose a value for t; leave it as "t").

<u>Exercise 2.</u> Show that if $B = C^{-1}AC$, then $B^k = C^{-1}A^kC$ for all $k \geq 0$. Deduce from this that for any polynomial p, $p(B) = C^{-1}p(A)C$. This result remains valid for any function defined by a convergent Taylor series: if $B = C^{-1}AC$, and if the series for $f(A)$ converges, then so does the series for $f(B)$, and $f(B) = C^{-1}f(A)C$.

<u>Exercise 3.</u> Suppose f is a function whose Taylor series $\sum_{k=0}^{\infty} b_k x^k$ converges for all x. Show that if B is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then for each k, B^k is a diagonal matrix with entries $\lambda_1^k, \ldots, \lambda_n^k$, and deduce from this that $f(B)$ is a diagonal matrix with diagonal entries $f(\lambda_1), \ldots, f(\lambda_k)$. (Thus f of a diagonal matrix is very easy to compute.)

In view of Exercises 2 and 3, $f(A)$ can be computed easily for any diagonalizable matrix A as follows. (1) Find an eigenbasis for A, and let C be the matrix whose columns are the eigenvectors. Then $B = C^{-1}AC$ is a diagonal matrix whose diagonal entries are the eigenvalues of A. Compute C^{-1} for later use. (2) Since $B = C^{-1}AC$, we have $A = CBC^{-1}$, and hence $f(A) = Cf(B)C^{-1}$ by Exercise 1 (extended to Taylor series). Compute $f(B)$ using the result of Exercise 2, then obtain $f(A)$ by multiplying by C and C^{-1}.

<u>Exercise 4.</u> Compute e^A for each of the following matrices. (a) $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$. (b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.