Sets and Functions

Sets

A *set* is a collection of objects called *elements*. Curly braces { } are used to display the list of elements explicitly or by description. For example,

$$\{1, 2, 15\}$$

is the set whose elements are the numbers 1, 2, and 15, while

$$\{\text{real numbers greater than } 5\}$$

is the indicated set of real numbers. Elements of a set can be numbers, vectors, frogs, or anything else.

Two sets are *equal* if they have exactly the same elements. Thus

$$\{\text{cat}, \text{dog}, \text{elephant}\} = \{\text{elephant}, \text{cat}, \text{dog}\}.$$

Sets can have a finitely many elements (as in the examples above) or an infinitely many elements (for example, the set of real numbers).

The empty set. It is also useful to have a notation for a set that contains no elements. This set is called the *empty set*, and is denoted $\emptyset$. One reason this is useful is that it lets one say "Let S be the set of objects with such-and-such property," without knowing ahead of time that there *are* any objects with this property.

Notation for elements.

The symbol "$\in$" is used to indicate that an object belongs to a particular set; "$\notin$" is used to indicate that an object does not belong to that set. Pronunciation of these symbols depends on the context, and often includes a verb (most commonly some form of the verb "to be"). For example,

- The sentence "If $x \in S$, then x is blue" is read "If x is an element of S [or simply: If x is in S,] then x is blue."
- In the sentence "If $x \notin S$, then x is odd," the phrase "If $x \notin S$ is read "If x is not in S," or "If x is not an element of S," or "If x does not lie in S."
- The phrase "If $x \in S$ is greater than 5" is read "If x, in S, is greater than 5." [1]

[1] As a rule, mathematical symbols do not incorporate punctuation; this is an allowed exception to that rule.

- The complete sentence "Let $x \in S$ [period]" is read "Let x be an element of S [period]" or "Let x be in S."

- The complete sentence "Let $x \notin S$ [period]" is read "Let x not be an element of S".

Notation for subsets.

We say that a set B is a subset of a set A if every element of B is also an element of A. We write $B \subset A$ or $B \subseteq A$ in this case.[2] (Read this as "B is a subset of A" or "B is contained in A").

Examples:

- Every set A has two "trivial" subsets: $A \subseteq A$ and $\emptyset \subseteq A$. (Of course these two subsets are the same if A itself is empty!)

- $\{$ positive real numbers $\} \subseteq \{$ all real numbers $\}$.

- The set $\{1, 2, 3\}$ has exactly the following eight subsets: $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.

  (Note that if $x \in S$, we don't write "$x \subseteq S$"; instead we write "$\{x\} \subseteq S$." An *element* of S is, in general, not itself a *subset* of S. Rather, an element of S determines a *singleton set* containing just that one element, and it is this singleton set that is a subset of S.)

As with the element-symbol, pronunciation of the subset-symbols depends on context, and sometimes includes a verb (most commonly some form of the verb "to be"). For example:

- In the sentence "If $B \subseteq A$, then B has at most five elements", the portion "If $B \subseteq A$" is read "If B is a subset of A." "If B is contained in A."

- The phrase "If $B \subseteq A$ is the set of all even integers", is read "If B subset A is the set of all even integers" (in this reading, "subset" is a *preposition*, like "in") or "If B, in A, is the set of all even integers."

[2]For most mathematicians the symbols "$\subseteq$" and "$\subset$" are synonymous, but some mathematicians—a minority—use "$\subset$" only for *proper* subsets; i.e. they exclude the case of equal sets. The motivation for the latter usage is to make these symbols analogous to the "$<$" and "$\leq$" symbols. As appealing as that motivation may be, *using "$\subset$" to mean "proper subset" is, at the present time, a bad idea* (at least in undergraduate classes and textbooks); the downside far outweighs any esthetic appeal. Using "$\subset$" to mean *proper* subset is very confusing, because historically, "$\subset$" meant "subset", not "proper subset". Students were even taught this in elementary school, and few people (if any) used "$\subset$" to mean "proper subset" until a few decades ago. "Classic" math textbooks—the ones many of your professors learned from, including me—in every field of mathematics, all used "$\subset$" to mean "subset", not "proper subset". I still mean "subset" whenever I use this symbol. Perhaps in 50 years it will be standard to use "$\subset$" to mean "proper subset", but in the meantime, it's probably best to avoid any possible misinterpretation by using "$\subseteq$" for (any) subset, and "$\subsetneq$" for a proper subset.

- The complete sentence "Let $B \subseteq A$ [period]" is read "Let B be a subset of A."

A common strategy for showing that two sets A and B are equal is to show that each set is a subset of the other. (In fact this is a more precise way to define "equal sets" than the way above. If two sets are infinite, what else would you mean by saying that they have the same elements?)

Set selector notation (also called "set builder notation")

When one wants to define a set by some properties of its elements, *set selector* notation is often used. Either a colon or a vertical bar may be used, as in

$$A = \{\text{doodad} : \text{doodad has property } C\}$$

and

$$A = \{\text{doodad} \mid \text{doodad has property } C\}.$$

In each case, the notation defines a set A. The vertical line and the colon are pronounced "such that". (These symbols *never* mean "equals".) Within the curly braces, the symbol that appears to the left of the colon (or vertical bar) is the notation for a typical element of A.

Examples.

- $\{x : x \text{ is a real number and } 0 < x < 5\}$.
- $\{x \mid x \text{ is a real number and } 0 < x < 5\}$.

The colon and vertical bar mean exactly the same thing, and are read exactly the same way; which one you use is a matter of personal preference. The linear algebra textbook by Friedberg, Insel, and Spence uses the colon. I use both: I grew up using the vertical bar, but in recent years have been transitioning to using the colon more. Both examples above are read "The set of x such that x is a real number and $0 < x < 5$."

It is common to use "$\mathbf{R}$" or "$\mathbb{R}$" to denote the set of real numbers, although some authors (including Friedberg, Insel, and Spence) simply use "R". I use $\mathbf{R}$ in printed or "e-printed" material (like this handout) and use $\mathbb{R}$ at the blackboard.[3]

An alternative way of describing the set above is $\{x \in \mathbf{R} \mid 0 < x < 5\}$. We use the notation $\mathbf{R}^n$ (which is read "R-n", not "R to the n") for the set of ordered n-tuples of real numbers. Thus

$$\begin{aligned} \mathbf{R}^2 &= \{(x, y) \mid x, y \in \mathbf{R}\}, \\ \mathbf{R}^3 &= \{(x, y, z) \mid x, y, z \in \mathbf{R}\}, \\ \mathbf{R}^n &= \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbf{R}\}, \end{aligned}$$

[3]These usages were standard when I was a student. Historically, $\mathbf{R}$ was preferred in printed materials; the "blackboard bold" symbol $\mathbb{R}$ was invented because you can't efficiently produce boldface with chalk, pen, or pencil.

etc. It is also common to write these n-tuples vertically (as column vectors) instead of horizontally.

More advanced examples of using set selector notation.

Sometimes additional, very broad restrictions on a set's elements are put to the left of the bar (or colon), as in

$$A = \{\text{persons } P : P \text{ is over 5 feet tall}\}.$$

This tells us that A is a set contained in the "universe" of persons (really just a larger set of which A is a subset), and that what determines whether a given person is in A is whether that person is over 5 feet tall.

Sometimes additional objects appear in the same sentence as "$A = \dots$". Objects that are defined or restricted only *outside* the curly braces are *constants* for the given set A. The notation for these constants (e.g. $a_1, \mathbf{v}_2$) is fixed by the notation outside the brackets; you must use the same letters throughout the problem, definition, theorem, proof, etc., to refer to the same set A. Different values of the constants may distinguish one set A from another, but are constants as far as a single set A is concerned. ("Constant" here does not necessarily mean "number".) It is irrelevant whether these constants appear *before* or *after* the pair of curly braces; all that matters is that they are *outside*.

- Example 1: $A = \{\text{persons } P : P \text{ is over } b \text{ feet tall}\}$, where b is a real number.

- Example 2: $A = \{\text{persons } P : P \text{ sometimes wears a } b\}$, where b is an article of clothing.

  In each of Examples 1 and 2, the restriction on b occurs outside the brackets, meaning that b is a constant within each set. In example 1, for $b = 5$, we get one specific set A; for $b = 6$, we get another specific set A. Within each of these sets, b is constant. In example 2, when b is a hat we get one specific set A; when b is a tie we get another specific set A. The definition of A thus allows us to consider a lot of sets at the same time, rather than forcing us to treat each possible A separately.

  When additional objects in the set-definition sentence occur *within* the curly brackets, to the right of the bar (or colon), the interpretation is different. This is most easily illustrated by examples before giving the general rules.

- Example 3: $A = \left\{ \mathbf{v} \in \mathbf{R}^3 : \mathbf{v} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ for some } c_1, c_2 \in \mathbf{R} \right\}.$

  In this example, c_1, c_2 are variable scalars (real numbers) that distinguish *one element* of the fixed set A from *another element*; for every choice of the pair (c_1, c_2), we get a different element of A.

- Example 4: Let $\mathcal{F}(\mathbf{R},\mathbf{R})$ be the set of functions with domain and codomain $\mathbf{R}$ (see the section "Functions", coming up shortly, for this terminology), and let

$$\begin{aligned} A \;=\; \{f \in \mathcal{F}(\mathbf{R},\mathbf{R}) : \text{there exist } c_1, c_2 \in \mathbf{R} \text{ such that } f \text{ is given by} \\ f(x) = c_1 \cos x + c_2 \sin x \quad \text{for each } x \in \mathbf{R}\}. \end{aligned}$$

  Again, c_1, c_2 are variable scalars that distinguish *one element* of the fixed set A from *another element*; for every choice of the pair (c_1, c_2), we get a different element of A. The variable x does not restrict f at all, or distinguish one f from another; x is simply a "dummy variable" we temporarily need in order to describe the relation between two functions.

  In general, when additional objects in the set-definition sentence occur *within* the curly braces, to the right of the colon (or vertical bar), and furthermore

  (i) potentially distinguish one element of A from another, and

  (ii) are defined or restricted only by words *within* the braces,

  then these objects are variables *within* the set A: different values of these variables determine different elements of A. In examples 3 and 4, the numbers c_1 and c_2 are such objects; in example 4, x is not.

- Example 5: $H = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\}$, where $a_1, a_2, \ldots, a_n$ are scalars (*i.e.* real numbers) that are not all zero.

  The notation tells us that there are many sets H defined by this notation, one set for each choice of the scalars. (Because n first appears to the left of the colon, we treat n as a constant for a given set H, just as if a definition of n [or restriction on n] were to occur outside the brackets.) So, for example, if $n = 2$ then one possible H is $H_1 = \{(x_1, x_2) \in \mathbf{R}^2 : 5x_1 + 3x_2 = 0\}$; another is $H_2 = \{(x_1, x_2) \in \mathbf{R}^2 : -7x_1 + 6x_2 = 0\}$. The vector $(1, -5/3)$ is in H_1 but not in H_2.

- Example 6: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be arbitrary vectors in a vector space V. Let

$$H = \{\mathbf{v} \in V : \mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \text{where } a_1, a_2, \ldots, a_n \text{ are all scalars}\}.$$

  In this example, each choice of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ leads to a *different set* H. For a *single* set H, we must treat the $\mathbf{v}_j$ as constants, not introducing any new notation for them. But within H, the scalars a_j are variables, whose choices determine different vectors in the same set H. These variables can be called by any names we want (e.g. $b_1, \ldots, b_n$).

Equations of a geometric object

If S is a "geometric object"—*e.g.* some subset of $\mathbf{R}^2$ or $\mathbf{R}^3$, such as a line, plane, or sphere—we sometimes are asked to "find an equation (or equations) for S". This means we are going to describe S by the property that a given point in $\mathbf{R}^2$ or $\mathbf{R}^3$ is an element of S if and only if the given equations are satisfied. Sometimes these equations are written in terms of the coordinates of Euclidean space (such as x, y, z); other times they are *parametric*, written in terms of variables that are not themselves coordinates of Euclidean space.

- Example 1 (non-parametric). *The plane P in $\mathbf{R}^3$ with equation $x + 2y + 3z = 6$* means
$$P = \{(x, y, z) \in \mathbf{R}^3 \mid x + 2y + 3z = 6\}.$$
- Example 2 (parametric). Let $\mathbf{r} = (x, y, z), \mathbf{u} = (-2, 1, 0), \mathbf{v} = (-3, 0, 1)$. *The plane P in $\mathbf{R}^3$ with equation $\mathbf{r} = (6, 0, 0) + t_1\mathbf{u} + t_2\mathbf{v}$* means
$$P = \{\mathbf{r} \in \mathbf{R}^3 \mid \text{there exist } t_1, t_2 \in \mathbf{R} \text{ such that } \mathbf{r} = (6, 0, 0) + t_1\mathbf{u} + t_2\mathbf{v}\}.$$

(In fact, the two planes described in the two example above are the same. Why?)

Intersection and union

The *intersection* of two sets A and B is the set of elements common to both A and B. The intersection is denoted $A \bigcap B$:
$$A \bigcap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The *union* of two sets A and B is the set of elements contained in A *or* B, (*i.e.* in at least one of the two; in math, "or" is always inclusive). The union is denoted $A \bigcup B$:
$$A \bigcup B = \{x \mid x \in A \text{ or } x \in B\}.$$
Note the correspondence between the symbols and the words:
$$\begin{array}{ccl} \bigcap & \longleftrightarrow & \text{and ,} \\ \bigcup & \longleftrightarrow & \text{or .} \end{array}$$

Functions

A *function* is an assignment of exactly one element of a specified set, called the *codomain*, to an element of another set, called the *domain*.[4] Thus to specify a function, one needs three pieces of information:

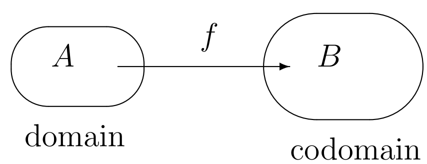
[4]More precisely: a function is an *ordered triple* $f = (A, B, C)$, where A is a set called the *domain* of f, B is a set called the *codomain* of f, and C is a subset of the Cartesian product $A \times B$ with the property that for each $a \in A$, there is a unique $b \in B$ for which $(a, b) \in C$. Two functions $f = (A, B, C)$ and $g = (A', B', C')$ are *equal* if and only if $A = A', B = B'$, and $C = C'$.

- the domain A
- the codomain B
- the "rule" specifying which element of B gets assigned to each element of A.

A function with domain A and codomain B is said to be a function *from A to B*. The mathematical notation for this is $f : A \to B$, which is read "f, from A to B".[5] For f to be a function from A to B, we require that $f(x)$ be defined for every $x \in A$, and that $f(x) \in B$ for every $x \in A$. We do not require that every $y \in B$ be equal to $f(x)$ for some $x \in A$.

It is often helpful to picture a function using a diagram like the one in Figure 1.

Figure 1: A function $f : A \to B$



Note that *codomain* does not mean the same thing as *range*.[6]

[5]Other readings are possible, depending on context, but we're not going into them here.

[6]Unfortunately, there are some modern textbooks, including the Abstract Algebra textbook by J. Gallian, that have co-opted the word "range", using it with a meaning that conflicts with its historical usage, as well as with (to my knowledge) most of contemporary mathematics. The historical meaning is the one given in these notes (and in *most* textbooks that I've seen that have a section on functions). For real-valued functions of a real variable, this meaning of "range" is exactly the usage you learned in calculus or pre-calculus; the meaning in (e.g.) Gallian's textbook is not. With the latter usage, the range of *every* function in precalculus and Calculus 1 is $\mathbf{R}$, the set of real numbers; everything you learned about "range" in (pre)calculus goes out the window. Adopting this different meaning of *range* for post-calculus students is akin to saying, "The animal that you've always called a *cat*, we're now going to call a *dog*, at least in this class." This causes additional problems when students take subsequent classes whose instructors expect students to have learned the *conventional* meanings of function-terminology.

Until fairly recently (relative to the history of function-terminology), although it had always been fairly common to use the word *domain* for the set A in "function from A to B," there was *no* common textbook terminology for the set B. In situations where mathematicians wanted a name for this set, they typically chose some word (e.g. "target") *that did not already have a* **conflicting** *meaning in mathematics.* In the 1970s, or perhaps earlier, some authors (e.g. the renowned Yale algebraist N. Jacobson, in *Basic Algebra I*, 1974) started using the word "codomain" for B—a word that, while not exactly beautiful, had the crucial feature of *not conflicting with pre-existing terminology.* It also had the advantage that, in the context of the definition of "function" the meaning of "codomain" was self-evident: of the two sets appearing in "function from A to B", the *codomain* was simply *the set that was* not *the domain.* In the 1990s, "codomain" took wider and wider hold, as instructors found that, as an aid to *teaching* what an abstract function is, it was convenient to have a standard

In Calculus 1 and precalculus, you saw functions with different ranges, the functions all had the same codomain: $\mathbf{R}$. (Hence it would have been confusing, at that level of your math education, to introduce to you a special word like *codomain.*) In Calculus 3, when you spoke of vector-valued functions, the codomain was $\mathbf{R}^3$ or $\mathbf{R}^2$. At any one time it was just one of these, so again there was no need to introduce a word for "codomain".

In linear algebra, the domain and codomain are usually vector spaces, and functions are often called *transformations, maps*, or *mappings*.

If $f : A \to B$ is a function, we say that f is a *B-valued function on A*. For example, if $B = \mathbf{R}$, we speak of real-valued functions. We commonly call $\mathbf{R}^n$-valued functions *vector-valued* functions whether $n = 2, 3$, or anything larger.

The *range* of a function f is set of elements of the codomain actually "hit" by f, *i.e.* the set of elements of the form $f(x)$ (thus the range is a *subset* of the codomain). A more pictorial synonym for range is *image*, which is used more often than *range* outside of calculus classes. Thus, if $f : A \to B$ is a function, then

$$\begin{aligned}\text{image}(f) = \text{range}(f) &= \{f(x) \mid x \in A\} \\ &= \{b \in B \mid b = f(x) \text{ for some } x \in A\}.\end{aligned}$$

The word "image" actually has a more general usage in the setting above: for any subset $C \subseteq A$, the *image of C under f* is the set

$$\begin{aligned}f(C) &:= \{f(x) \mid x \in C\} \\ &= \{b \in B \mid b = f(x) \text{ for some } x \in C\}.\end{aligned}$$

Thus *the image of* $f : A \to B$ is the same as *the image of the whole domain A under* f, and

$$\text{image}(f) = \text{range}(f) = f(A).$$

As we move into higher mathematics, it becomes increasingly important to be able to think of a function as an *object* (so that for instance one can talk about *sets of functions*), and to understand that in Calculus-1 notation like "$f(x)$", the function is $\underline{f}$, **NOT** $f(x)$. The letter used for the *domain* variable is a *dummy variable*; it is not part of the name of the function, and can be any letter that doesn't already have a pre-assigned meaning. For example, if $f : \mathbf{R} \to \mathbf{R}$ is the function defined by $f(x) = x^2 + 2x + 3$, then $f(u) = u^2 + 2u + 3$, $f(y) = y^2 + 2y + 3$, etc.

For this a different picture that is often useful is one in which a function $f : A \to B$ is viewed as a machine whose input is elements of A and whose output is elements of B:

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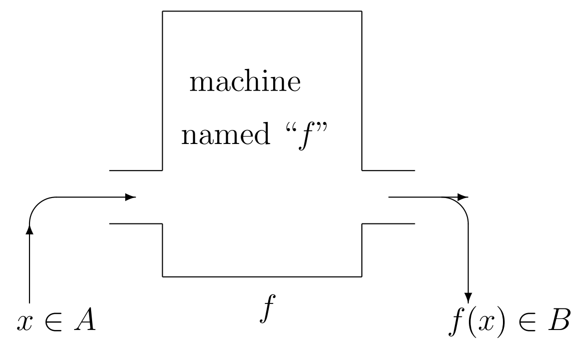
name for this set. (I had used "target" up till that time—and correspondingly, would often use "source" instead of "domain"—but switched from "target" to "codomain" because that was the terminology that students entering my advanced courses had learned in prerequisites, and because, unlike "range", "codomain" was a *reasonable, <u>non-conflicting</u>* name for this set.) To my knowledge, "codomain" has become, far and away, the standard term nowadays. Indeed, until 2021, I had never heard of *any* mathematician using "range" for this set.

Figure 2: A function $f : A \to B$



While this picture can be quite useful, it is not perfect: it understates (and is even misleading about) the importance of the sets A and B. By definition, *changing either of these sets changes the function.*

But an advantage of this picture is that it allows us to view some very complicated functions in more simple terms, avoiding some mental baggage. For example, let $S = \{f : \mathbf{R} \to \mathbf{R} : f \text{ is infinitely differentiable}\}$, the set of infinitely differentiable real-valued functions with domain $\mathbf{R}$. Define a function $D : S \to S$ by $D(f) = f'$ (the derivative of f). In your differential equations class, you called (or will call) D an *differential operator*. Our more abstract picture shows us that a differential operator is just another function—it simply has a domain and codomain that are different from the ones we're more used to. The domain and codomain of D are *sets of functions* $f : \mathbf{R} \to \mathbf{R}$, not subsets of $\mathbf{R}$ or $\mathbf{R}^n$ at all.

One-to-one and onto

A function $f : A \to B$ is called *one-to-one* or *injective* if f takes no two elements of A to the same element of B, *i.e.* if $f(x) \neq f(y)$ whenever $x \neq y$. *One-to-one* is often abbreviated as 1-1.

A function $f : A \to B$ is called *onto* or *surjective* if every element of B is "hit" by f (*i.e.* is in the image of f). Put another way, $f : A \to B$ is onto if $\text{image}(f) = B$.

If f is both 1-1 and onto, f is called bijective.

Functions that are injective, surjective, or bijective respectively are called injections, surjections, and bijections.

Inverse functions

Given a function $f : A \to B$, we say that f has an *inverse* (or *inverse function*), or is *invertible*, if there is a function $g : B \to A$ satisfying

$$\begin{array}{rrcll} & g(f(a)) & = & a & \text{for all } a \in A \quad (1) \\ \text{and} & f(g(b)) & = & b & \text{for all } b \in B. \quad (2) \end{array}$$

A function g satisfying (1)–(2) is called an *inverse of f* (or *inverse function*, with "of f" being understood).

It is not hard to show that, for any function f,

f has an inverse function if and only if f is bijective.

When $f : A \to B$ *does* have an inverse function (thus, when f is bijective), it has *exactly one* inverse function (i.e. f has a *unique* inverse). This is easily demonstrated: if $g_1, g_2 : B \to A$ satisfy conditions the conditions for g in (1)–(2), then for all $b \in B$,

$$\begin{array}{rcll} g_2(b) & = & g_2(f(g_1(b))) & \text{(using (2) with } g = g_1) \\ & = & g_1(b) & \text{(using (1) with } g = g_2 \text{ and } a = g_1(b)); \end{array}$$

i.e. $g_2 = g_1$. Thus, when f is invertible, we can unambiguously speak of *the* inverse of f, and introduce notation for it: f^{-1}.

Difference between a function and a formula

In your calculus classes, you commonly referred to things like $\sin x, e^x, x^2 + 3x + 2$, and $\sqrt{x}$ as functions. Technically this is wrong: these are *formulas,* or *expressions*, rather than functions. They are similar to the "machine" part of Figure 2, in which the importance of stating the domain and codomain is suppressed. This does not mean that everything you learned in calculus is wrong! In your calculus classes, you implicitly used two principles: (i) the codomain of *every* function was $\mathbf{R}$, so there was never any need for an explicit, new word "codomain", and (ii) for the domain of the function, you used the "implied domain" of the formula: the largest set of real numbers for which the formula made sense. For example, in "$\sqrt{x}$", it was always assumed that x was a real number, and the domain was the inteval $[0, \infty) = \{x \in \mathbf{R} \mid x \geq 0\}$. This is an example of "abuse of terminology" (a cousin of "abuse of notation"): terminology that is technically incorrect or imprecise, but (in this case) is being used because (i) the correct, precise terminology might be cumbersome and distracting (or perhaps we're just too lazy to write it!), and (ii) the terminology being used doesn't have *another* meaning that conflicts with the intended one. In place of writing "the function $\sqrt{x}$" or "the function $f(x) = \sqrt{x}$" one could more correctly (but more lengthily) write "the function $f : [0, \infty) \to \mathbf{R}$ defined by $f(x) = \sqrt{x}$."

Abusing terminology is not always a bad thing, *provided that both the writer and reader know what is meant* without any mind-reading, and that there is some advantage (e.g. time-saving) to using the incorrect or imprecise terminology. You will probably see most of your instructors, and the authors of your calculus and differential equations textbooks, do this when they write down functions.

For commonly used functions, one way to avoid this abuse of terminology is to give these functions a permanent name. For example, "sin" and "cos" are functions; *by definition* their domains and codomains are $\mathbf{R}$. (But $\sin(x), \cos(x)$ are technically not functions—they are merely the *output* of the machine in Figure 2.) Another example is the exponential function $\exp : \mathbf{R} \to \mathbf{R}$, which is defined by $\exp(x) = e^x$.

The importance of distinguishing between functions and formulas becomes clearer when one thinks about inverse functions. As mentioned earlier, Therefore *the sine function has no inverse*! The function arcsin or $\sin^{-1}$ is *not* the inverse of sine. Instead, it is the inverse of the function

$$\begin{aligned} \text{Sine} : [-\pi/2, \pi/2] &\to [-1, 1], \quad \text{given by} \\ \text{Sine}(x) &= \sin x. \end{aligned}$$

(I've used a capital S to distinguish Sine from sine.) The functions Sine and sine have the same formula, but are different functions: both the domain and codomain of Sine are different from those of sine. Similarly, the function sqrt: $[0, \infty) \to \mathbf{R}$ given by $\text{sqrt}(x) = \sqrt{x}$ has no inverse because it is not onto. However, the function Sqrt:$[0, \infty) \to [0, \infty)$ defined using the same formula does have an inverse.